Q1. Solve $2y'' + 6y' + 4y = \cos(e^x)$.

The standard form of the equation is: $y'' + 3y' + 2y = \frac{1}{2}\cos(e^x)$

The corresponding homogeneous equation $y''+3y'+2y=0$ has auxiliary equation: $m^2+3m+2=0$.

The roots of the auxiliary equation are $m_1=-1$ and $m_2=-2$

The complementary solution is: $y_c = c_1 e^{-x} + c_2 e^{-2x}$

The Wronskian of e^{-x} and e^{-2x} is given by:

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

We shall use the method of "variation of parameters" to find a particular solution y_p:

$y_p = u_1 e^{-x} + u_2 e^{-2x}$, where:

$$u_1' = -\frac{e^{-2x}\cos(e^x)}{-2e^{-3x}} = \frac{1}{2}e^x\cos(e^x),$$

and

$$u_2' = \frac{e^{-x}\cos(e^x)}{-2e^{-3x}} = -\frac{1}{2}e^{2x}\cos(e^x)$$

Thus. $u_1 = \frac{1}{2}\int e^x\cos(e^x)\,dx = \frac{1}{2}\sin(e^x)$, and

$$u_2 = -\frac{1}{2}\int e^{2x}\cos(e^x)\,dx = -\frac{1}{2}\left[e^x\sin(e^x) - \int e^x\sin(e^x)\,dx\right]$$

$$= -\frac{1}{2}\left[e^x\sin(e^x) + \cos(e^x)\right].$$

Hence: $$y_p = \frac{e^{-x}}{2}\sin(e^x) - \frac{e^{-2x}}{2}\left[e^x\sin(e^x) + \cos(e^x)\right]$$

$$= -\frac{e^{-2x}}{2}\cos(e^x).$$

Therefore, the general solution is: $y = y_c + y_p = c_1e^{-x} + c_2e^{-2x} - \frac{1}{2}e^{-2x}\cos(e^x)$.

Q2. Solve the initial-value problem

$$y''' - y'' + y' = 0,\ y(0) = y'(0) = y''(0) = \sqrt{3}.$$

The associated auxiliary equation is: $m^3 - m^2 + m = 0$ or $m(m^2 - m + 1) = 0$. Its roots are given by:

$$m_1 = 0,\ m_2 = \frac{1 - i\sqrt{3}}{2},\ m_3 = \frac{1 + i\sqrt{3}}{2}$$

The solution of this "IVP" is of the form:

$$y = C_1 + e^{\frac{1}{2}x}\left[C_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right].$$

- $y(0) = C_1 + C_2 = \sqrt{3}$ (1)
- $y' = \frac{1}{2}e^{x/2}\left[C_2\cos\left(\frac{\sqrt{3}}{2}x\right) + C_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{x/2}\left[-C_2\frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}x\right) + C_3\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}x\right)\right]$

So $y'(0) = \frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 = \sqrt{3}$ (2)

- $y'' = \frac{1}{4}e^{x/2}\left[C_2\cos\left(\frac{\sqrt{3}}{2}x\right) + C_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + \frac{e^{x/2}}{2}\left[-C_2\frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}x\right) + C_3\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}x\right)\right]$
$+ \frac{e^{x/2}}{2}\left[-C_2\frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}x\right) + C_3\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{x/2}\left[-C_2\frac{3}{4}\cos\left(\frac{\sqrt{3}}{2}x\right) - C_3\frac{3}{4}\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

So $y''(0) = \frac{C_2}{4} + \frac{\sqrt{3}}{4}C_3 + \frac{\sqrt{3}}{4}C_3 - \frac{3}{4}C_2 = -\frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 = \sqrt{3}$ (3)

Now: "(2)+(3)" implies: $\sqrt{3}C_3 = 2\sqrt{3}$ i.e $\boxed{C_3 = 2}$

(2) implies: $\boxed{C_2 = 0}$

(1) implies: $\boxed{C_1 = \sqrt{3}}$

Hence, the solution is given by:

$$y = \sqrt{3} + 2e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

Q3. Find a particular solution for

$$y'' - 4y = e^{2x} + \cos x.$$

The complementary solution is $y_c = c_1 e^{-2x} + c_2 e^{2x}$

$(D-2)$ is an annihilator of e^{2x}

(D^2+1) is an annihilator of $\cos x$

<u>Now</u>. $(D-2)(D^2+1)[y''-4y] = (D-2)(D^2+1)(e^{2x}+\cos x) = 0$

The auxiliary equation of the above "new diff. equ" is:

$(m-2)(m^2+1)(m^2-4) = 0$ or $(m-2)^2(m+2)(m^2+1) = 0$

The roots are: $\underset{\text{double}}{m_1 = 2}$, $m_2 = -2$ and $m_3 = \pm i$

The general solution is of the form:

$$y = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x} + c_4 \cos x + c_5 \sin x$$

$$= y_c + c_2 x e^{2x} + c_4 \cos x + c_5 \sin x$$

Thus y_p is of the form: $y_p = c_2 x e^{2x} + c_4 \cos x + c_5 \sin x$

Substituting y_p into the initial equation implies:

$$-5c_4 \cos x - 5c_5 \sin x + 4c_2 e^{2x} = e^{2x} + \cos x$$

Therefore, we must have: $c_2 = \frac{1}{4}$, $c_4 = -\frac{1}{5}$ and $c_5 = 0$

Hence $y_p = \frac{1}{4} x e^{2x} - \frac{1}{5} \cos x$

Q4. Use the substitution $x = e^t$ to transform the equation

$$x^2y'' + \frac{1}{4}y = \ln x$$

to a differential equation with constant coefficients, and then solve it.

We have that: $\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$ and $\frac{d^2y}{dx^2} = \frac{1}{x^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$.

So the differential equation becomes:

(*) $\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{1}{4}y = t$ Its corresponding auxiliary equation is: $m^2 - m + \frac{1}{4} = 0$ or $(m - \frac{1}{2})^2 = 0$.

The complementary solution is: $y_c = (c_1 + c_2 t)e^{t/2}$

Now we try a particular solution y_p of the form:

$$y_p = A + Bt$$

We substitute y_p into (*) to find A and B:

$$-B + \frac{1}{4}A + \frac{1}{4}Bt = t$$

Thus $A = 16$ and $B = 4$

Therefore, the general solution of the original equation is:

$$y = y_c + y_p = (c_1 + c_2 t)e^{t/2} + 16 + 4t$$

$$= c_1\sqrt{x} + c_2\sqrt{x}\ln x + 16 + 4\ln x$$

Q5. Which of the following sets of functions is linearly dependent on $(-\infty, +\infty)$?

(a) $\{5x, 2\sin^2 x, 3\cos^2 x\}$

(b) $\{\cos^2 x, \sin^2 x, \sin 2x\}$

(c) $\{\sec^2 x, \tan^2 x, x\}$

→(d) $\{(x+1)^2, (x-1)^2, 3x\}$

(e) $\{(x+2)^2, (x+1)^2, 2x\}$

It is easy to see that:

$$\frac{3}{4}(x+1)^2 - \frac{3}{4}(x-1)^2 = \frac{3}{4}\left[(x+1)^2 - (x-1)^2\right] = \frac{3}{4}(x+1-x+1)(x+1+x-1)$$

$$= \frac{3}{4} \cdot 2x \cdot 2 = 3x$$

Thus $\{(x+1)^2, (x-1)^2, 3x\}$ is linearly dependent

Clearly all the other sets are linearly independent.

Q6. Let $f(x) = x\sin(2x) + x^2e^{-3x}\cos x$. Which of the following differential operators is an annihilator of $f(x)$?

(a) $5(D^2 - 6D + 10)^3(D^2 - 4)$

→ (b) $3(D^2 + 6D + 10)^5(D^2 + 4)^4$

(c) $-2(D^2 - 6D + 10)^3(D + 4)^2$

(d) $(D^2 + 6D + 10)^2(D^2 + 4)^2$

(e) $(D^2 + 6D + 10)^4(D^2 - 4)^2$

•) $(D^2+4)^2$ is an annihilator of lowest possible order of $x\sin(2x)$

•) $(D^2+6D+10)^3$ is an annihilator of $x^2e^{-3x}\cos x$ of lowest possible order

Q7. Given that $y_1(x) = x$ is a solution of

$$x^2y'' - x(x+2)y' + (x+2)y = 0,$$

a second solution linearly independant of $y_1(x)$ could be

(a) x^2e^x

(b) $2x$

(c) xe^{2x}

(d) $(3x-2)e^x$

→ (e) $4x(5-2e^x)$

The standard form of the equation is:

$$y'' - \left(\frac{x+2}{x}\right)y' + \frac{x+2}{x}y = 0$$

A second solution y_2 linearly independent of y_1 is:

$$y_2(x) = y_1(x)\int \frac{e^{-\int -\left(\frac{x+2}{x}\right)dx}}{y_1^2(x)}\,dx = x\int \frac{e^{\int\left(1+\frac{2}{x}\right)dx}}{x^2}\,dx$$

$$= x\int \frac{e^{x+\ln x^2}}{x^2}\,dx = x\int e^x\,dx = xe^x$$

<u>Now</u>: $4x(5-2e^x) = 20x - 8xe^x = 20y_1(x) - 8y_2(x)$ is a solution, linearly independent of $y_1(x)$.

Q8. A particular solution of

$$x^2y'' + xy' - y = \frac{x}{1-x}$$

could be

(a) $-\frac{1}{2}x^{-1}\ln\left|\frac{x}{1-x}\right| + \frac{1}{2}(x + \ln|1-x|)$

(b) $-\frac{1}{2}x\ln\left|\frac{x}{1-x}\right| + \frac{1}{2}(1 + x^{-1}\ln|1-x|)$

(c) $-\frac{1}{2}x\ln|x| + \frac{1}{2}(1 + x^{-1}\ln|1-x|)$

(d) $-\frac{1}{2}x\ln|1-x| + \frac{1}{2}(x^{-1} + \ln|1-x|)$

(e) $-\frac{1}{2}\ln\left|\frac{x}{1-x}\right| + \frac{1}{2}(1 + x\ln|1-x|)$

•) The complementary solution is: $y_c = c_1 x + c_2 x^{-1}$

•) The standard form of the equation is: $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x(1-x)}$

•) The Wronskian: $W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1}$,

$W_1 = \begin{vmatrix} 0 & x^{-1} \\ \frac{1}{x(1-x)} & -x^{-2} \end{vmatrix} = \frac{1}{x^2(1-x)}$, $W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{1}{x(1-x)} \end{vmatrix} = \frac{1}{1-x}$

•) $u_1' = \frac{W_1}{W} = -\frac{1}{2}\frac{1}{x(1-x)} \Rightarrow u_1 = -\frac{1}{2}\int \frac{1}{x(1-x)}dx = -\frac{1}{2}\left[\ln|x| - \ln|1-x|\right]$

•) $u_2' = \frac{W_2}{W} = -\frac{1}{2}\frac{x}{1-x} \Rightarrow u_2 = -\frac{1}{2}\int\frac{x}{1-x}dx = \frac{1}{2}\left[x + \ln|1-x|\right]$

•) $y_p = -\frac{1}{2}x\ln\left|\frac{x}{1-x}\right| + \frac{1}{2}x^{-1}\left[x + \ln|1-x|\right]$

$= -\frac{1}{2}x\ln\left|\frac{x}{1-x}\right| + \frac{1}{2}\left(1 + x^{-1}\ln|1-x|\right)$

Q9. Which of the following functions is a solution to

$$x^3y''' - xy' + 5y = 0$$

on $(0, +\infty)$.

→ (a) $\frac{3}{x} - 7x^2\sin(\ln x)$

(b) $-\frac{5}{x} + 13x^3\cos(\ln x)$

(c) $8x^2\cos(\ln x) - 19x^3\sin(\ln x)$

(d) $\frac{9}{x} + 11x^3\cos(\ln x) - 6x^2\sin(\ln x)$

(e) $15x^3\cos(\ln x) - 16x^3\sin(\ln x)$

This Cauchy-Euler equation has auxiliary equation:

$m(m-1)(m-2) - m + 5 = 0$ i.e $m^3 - 3m^2 + m + 5 = 0$

i.e $(m+1)(m^2 - 4m + 5) = 0$.

The roots are: $m_1 = -1$, $m_2 = 2+i$ and $m_3 = 2-i$

The general solution is:

$$y = \frac{c_1}{x} + c_2 x^2\cos(\ln x) + c_3 x^2\sin(\ln x) \quad , \quad x > 0.$$

"Semester 123"

MCQ Key Solution_ Exam 2, Math202

	Code 1	Code 2	Code 3	Code 4
Q5	(d)	(a)	(c)	(a)
Q6	(b)	(a)	(d)	(b)
Q7	(e)	(c)	(d)	(b)
Q8	(b)	(e)	(e)	(d)
Q9	(b)	(e)	(c)	(c)