FINAL EXAM – MATH 202 (Term 123)

July 28, 2013

Duration: **180** Minutes

Name: ______________________ **ID#:** ____________

Section/Instructor: ______________ **Serial #:** __________

Question #	Marks	Maximum Marks
Q1		17
Q2		16
Q3		14
Q4		15
Q5		15
Question #	Your Answer	Marks 0 or 7
Q6		
Q7		
Q8		
Q9		
Q10		
Q11		
Q12		
Q13		
Q14		
Total		**140**

Q1. Find two linearly independent series solutions of $2xy'' + y' + y = 0$, about the singular point $x = 0$. For each series, give only the three first terms.

We put $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ Then, we obtain.

$$2xy'' + y' + y = (2r^2 - r)c_0 x^{r-1} + \sum_{k=1}^{\infty} \left[2(k+r)(k+r-1)c_k + (k+r)c_k + c_{k-1}\right] x^{k+r-1} = 0$$

This gives the indicial equation $r(2r-1) = 0$, and the recurrence relations: $c_k = -\dfrac{c_{k-1}}{k(2k-1)}$ for $r = 0$, and $c_k = -\dfrac{c_{k-1}}{k(2k+1)}$ for $r = \frac{1}{2}$

$r = 0$ gives $c_1 = -c_0$, $c_2 = \frac{1}{6} c_0, \ldots$

and $r = \frac{1}{2}$ gives $c_1 = -\frac{1}{3} c_0$, $c_2 = \frac{1}{30} c_0, \ldots$

We therefore get two linearly independent series solutions.

$$y_1 = 1 - x + \frac{1}{6} x^2 + \cdots$$

and $$y_2 = x^{\frac{1}{2}} \left(1 - \frac{1}{3} x + \frac{1}{30} x^2 + \cdots\right)$$

Q2. Solve the linear system

$$X'(t) = AX(t) + F(t), \text{ where } A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \text{ and } F(t) = \begin{pmatrix} t \\ e^t \end{pmatrix}.$$

$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)$

The eingenvalues of A are : $\lambda_1 = 1$ and $\lambda_2 = 2$.

·) Let $K_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ be s.t $AK_1 = \lambda_1 K_1$ Then $\begin{cases} a_1 = a_1 \\ 2a_1 + 2b_1 = b_1 \end{cases}$ i.e

$K_1 = \begin{pmatrix} a_1 \\ -2a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, we choose $a_1 = 1$ and so $K_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector associated to λ_1

·) Let $K_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ be s.t $AK_2 = \lambda_2 K_2$ Then $\begin{cases} a_2 = 2a_2 \\ 2a_2 + 2b_2 = 2b_2 \end{cases}$ i.e $a_2 = 0$

Thus $K_2 = \begin{pmatrix} 0 \\ b_2 \end{pmatrix} = b_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ We might choose $b_2 = 1$, and so $K_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector associate to λ_2.

Therefore $X_c = \begin{pmatrix} e^t & 0 \\ -2e^t & e^{2t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ is the general solution of the homogeneous System.

Let the particular solution be of the form : $X_p = \begin{pmatrix} e^t & 0 \\ -2e^t & e^{2t} \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ Then

$X_p = \varphi(t) \int \varphi^{-1}(t) F(t) dt$, where $\varphi^{-1}(t) = \begin{pmatrix} e^{-t} & 0 \\ 2e^{-2t} & e^{-2t} \end{pmatrix}$. Thus

$X_p = \varphi(t) \int \begin{pmatrix} te^{-t} \\ 2te^{-2t} + e^{-t} \end{pmatrix} dt = \varphi(t) \begin{pmatrix} -te^{-t} - e^{-t} \\ -te^{-2t} - \frac{1}{2}e^{-2t} - e^{-t} \end{pmatrix}$. Hence :

$X_p = \begin{pmatrix} -t - 1 \\ t + \frac{3}{2} - e^t \end{pmatrix}$

Finally the general solution is :

$X = X_c + X_p = \begin{pmatrix} e^t & 0 \\ -2e^t & e^{2t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} -t - 1 \\ t + \frac{3}{2} - e^t \end{pmatrix} = \begin{pmatrix} C_1 e^t - t - 1 \\ -(2C_1 + 1)e^t + C_2 e^{2t} + t + \frac{3}{2} \end{pmatrix}$

Q3. Consider the matrix $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \end{pmatrix}$

a) Find e^{At}.

We have : $A^2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus

$$e^{At} = I_3 + At + A^2 \frac{t^2}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} t & -2t & t \\ 0 & 0 & 0 \\ -t & 4t & -t \end{pmatrix} + \begin{pmatrix} 0 & t^2 & 0 \\ 0 & 0 & 0 \\ 0 & -t^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+t & t^2-2t & t \\ 0 & 1 & 0 \\ -t & 4t-t^2 & 1-t \end{pmatrix}$$

b) Use e^{At} to solve the initial-value problem

$$X'(t) = AX(t), \ X(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ where } X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

$$X(t) = e^{At} C$$

$$= \begin{pmatrix} 1+t & t^2-2t & t \\ 0 & 1 & 0 \\ -t & 4t-t^2 & 1-t \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = C_1 \begin{pmatrix} 1+t \\ 0 \\ -t \end{pmatrix} + C_2 \begin{pmatrix} t^2-2t \\ 1 \\ 4t-t^2 \end{pmatrix} + C_3 \begin{pmatrix} t \\ 0 \\ 1-t \end{pmatrix}$$

$$X(1) = \begin{pmatrix} 2C_1 - C_2 + C_3 \\ C_2 \\ -C_1 + 3C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow C_2 = 1, \ C_1 = 2, \ C_3 = -2$$

The solution is :

$$X(t) = \begin{pmatrix} t^2 - 2t + 2 \\ 1 \\ -t^2 + 4t - 2 \end{pmatrix}$$

Q4. Solve the initial-value problem

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = -4x - 3y \\ x(0) = 1,\ y(0) = 2. \end{cases}$$

The system can be written as $X' = AX$, where $A = \begin{pmatrix} 1 & 2 \\ -4 & -3 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 2 \\ -4 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5$

Solving $\lambda^2 + 2\lambda + 5 = 0$ implies: $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$

Let $K = \begin{pmatrix} a \\ b \end{pmatrix}$ be s.t.: $(A - \lambda I_2)K = \begin{pmatrix} 2-2i & 2 \\ -4 & -2-2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or

$(2-2i)a + 2b = 0$. Choosing $a = 1$ gives $b = i - 1$. So, the eigenvector

is $K = \begin{pmatrix} 1 \\ i-1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{Re(K)} + i\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{Im\,K}$

The general solution. $X = c_1 e^{t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin 2t\right]$

$\qquad + c_2 e^{-t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 2t\right]$

Since $x(0) = 1$, $y(0) = 2$, we get: $c_1 = 1$, $c_2 = 3$

Therefore, the solution of the IVP is:

$$X = e^{-t}\begin{pmatrix} \cos 2t + 3\sin 2t \\ 2\cos 2t - 4\sin 2t \end{pmatrix}$$

Q5. Find the general solution of the system

$$X'(t) = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix} X(t).$$

The characteristic polynomial is:

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 1 & 2-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(6-5\lambda+\lambda^2-2)-(1-\lambda)+(2\lambda-2)$$
$$= (2-\lambda)(\lambda-1)(\lambda-4)+\lambda-1+2(\lambda-1) = -(\lambda-1)^2(\lambda-5)$$

The eigenvalues are $\lambda_1=\lambda_2=1$ and $\lambda_3=5$

Eigenvectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for the first eigenvalue satisfy $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e they have the form $\begin{pmatrix} -y-2z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ We get two linearly independent eigenvectors $K_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $K_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

The eigenvalue $\lambda = 5$ has eigenvectors of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfying $\begin{pmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. By elementary row operations (or otherwise), we get $\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ i.e $x=y=z$ and we can choose $K_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Hence the general solution of the system is:

$$X = c_1\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^t + c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t}$$

Q6. A solution of the equation $y''' - 4y'' + 4y' = 0$ could be

(a) $2xe^{2x} - 1$

(b) $x^2e^{2x} + 4$

(c) $7 - 3x$

(d) $5 - x^2e^x$

(e) $3 - xe^x$

The auxiliary equation is: $m^3 - 4m^2 + 4m = 0 \Leftrightarrow m(m-2)^2 = 0$

$\Leftrightarrow$ $m = 0$, $m = 2$ (root of order 2)

$Y = C_1 + (C_2 + C_3 x)e^{2x}$ is the general solution.

It is easy to see that:

$2xe^{2x} - 1 = C_1 + (C_2 + C_3 x)e^{2x}$, with $C_1 = -1$, $C_2 = 0$

and $C_3 = 2$.

Q7. If $y(x)$ is the solution of the initial-value problem

$$x^2y'' - 3xy' + 4y = 0,\ y(1) = 1,\ y'(1) = 0,$$

then $y(e)$ is equal to

(a) e^2

(b) $-e^2$

(c) $2e^2$

(d) $-2e^2$

(e) 0

The auxiliary equation is: $m(m-1)-3m+4=0$. The root $m=2$ is of order 2. So the general solution is:

$y = c_1x^2 + c_2x^2 \ln x$

Since $y(1)=1$, $c_1 = 1$.

$y' = 2c_1x + 2c_2 x \ln x + c_2 x$. So $y'(1) = 1$ implies $2c_1 + c_2 = 0$ i.e $c_2 = -2$.

Therefore: $y = x^2 - 2x^2 \ln x$, and hence $y(e) = -e^2$

Q8. The differential equation

$$mx^2ye^y - 2\cos y + (x^3ye^y + x^3e^y + nx\sin y)y' = 0$$

is exact when $m + n$ equals

(a) 2

→ (b) 5

(c) 1

(d) 4

(e) 3

Writing the DE as $Mdx + Ndy = 0$, we get:

$M = mx^2ye^y - 2\cos y$ and $N = x^3ye^y + x^3e^y + nx\sin y$.

DE is exact if $M_y = N_x$ i.e. $mx^2(1+y)e^y + 2\sin y = 3x^2ye^y + 3x^2e^y + n\sin y$

This holds iff $m=3$, $n=2$. So $m+n=5$

Q9. A glass of water initially at 70°F is placed in a freezer. The freezer is kept at a constant temperature of 50°F. After one hour the temperature of the water is 60°F. The time (in hour) needed for the water to reach 52.5°F after it is placed in the freezer is

(a) 2

→ (b) 3

(c) 1.5

(d) 2.5

(e) 1.75

$T(t) = c e^{kt} + T_m = c e^{kt} + 50$

$T(0) = C + 50 = 70 \Leftrightarrow C = 20$

$T(1) = c e^{k} + 50 = 60 \Leftrightarrow 20e^{k} = 10$

$\Leftrightarrow k = \ln\left(\frac{1}{2}\right) = -\ln 2.$

Let t^* be the required time. We have:

$T(t^*) = 20\, e^{-t^*\ln 2} + 50 = 52.5$

$\Leftrightarrow 20\, e^{-t^*\ln 2} = 2.5$

$\Leftrightarrow -t^*\ln 2 = \ln\left(\frac{1}{8}\right)$

$\Leftrightarrow t^* = \frac{-3\ln 2}{-\ln 2} = 3$

The time needed is 3 hours

Q10. If $y(x)$ is the solution of the initial-value problem

$$xy' - y + e^{\frac{1}{x}} = 0, \; y(1) = 0,$$

then $y(\frac{1}{\pi})$ equals

(a) $\dfrac{e}{\pi}$

→ (b) $\dfrac{e^{\pi} - e}{\pi}$

(c) $\pi e^{\frac{1}{\pi}}$

(d) $\dfrac{e^{\pi}}{\pi}$

(e) $\pi(e^{\frac{1}{\pi}} - e)$

The standard form of equation is:

$$y' - \frac{1}{x}y = -\frac{1}{x}e^{1/x}. \text{ Thus } P(x) = -\frac{1}{x}.$$

The integrator vector $:= e^{\int P(x)dx} = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$

Thus: $\dfrac{d}{dx}\left(\dfrac{y}{x}\right) = -\dfrac{1}{x^2}e^{1/x}$

$$\Rightarrow \frac{y}{x} = -\int \frac{1}{x^2} e^{1/x}dx = e^{1/x} + C$$

$$\Rightarrow y = x\left(e^{1/x} + c\right)$$

Now: $y(1) = e + C = 0$ i.e $C = -e$

Hence: $y = x(e^{1/x} - e)$ and therefore: $y\left(\frac{1}{\pi}\right) = \frac{1}{\pi}(e^{\pi} - e)$.

Q11. Given that $y_1(x) = x^2$ $(x > 0)$ is a solution of $(x^2 + x)y'' - xy' - \frac{2}{x}y = 0$, the solution of the initial-value problem

$$(x^2 + x)y'' - xy' - \frac{2}{x}y = 0,\ y(1) = 4,\ y'(1) = -4$$

satisfies $y(2)$ equals

(a) -2
(b) -4
(c) 2
(d) 4
(e) 0

A 2nd solution of the DE is $y_2 = x^2 \int \frac{e^{\int \frac{x\,dx}{x^2+x}}}{x^4}\,dx = -\frac{1}{6}\left(3 + \frac{2}{x}\right)$

So the general solution (for $x>0$) is:

$y = c_1 x^2 + c_2\left(3 + \frac{2}{x}\right)$

Since $y(1) = 4$, gives: $c_1 + 5c_2 = 4$.

Since: $y' = 2c_1 x - 2\frac{c_2}{x^2}$, $y'(1) = -4$ gives: $2c_1 - 2c_2 = -4$

Hence: $c_1 = -1$, $c_2 = 1$ i.e. $y = -x^2 + \left(3 + \frac{2}{x}\right)$ and $y(2) = 0$.

Q12. An annihilator of the function

$$f(x) = 5x^3e^{-2x}\cos 5x - 2e^{-2x}\sin 5x - 7x^2e^{-4x}$$

could be

(a) $(D+4)^2(D^2-4D+29)^4$

→ (b) $(D+4)^5(D^2+4D+29)^6$

(c) $(D^2-4D+29)^2(D-4)^3$

(d) $(D^2+4D+29)^3(D+4)^2$

(e) $(D-4)^2(D^2+4D+29)^4$

$(D^2+4D+29)^4$ is an annihilator of $5x^3e^{-2x}\cos 5x - 2e^{-2x}\sin 5x$.

$(D+4)^3$ is an annihilator of $-7x^2e^{-4x}$.

Q13. A solution of the equation

$$xy' + 2y = 4x^4y^4, \ x > 0,$$

could be

(a) $y^3 = \dfrac{1}{x^4}$

(b) $y^3 = \dfrac{1}{x^2(6+3x^2)}$

(c) $y^3 = \dfrac{1}{x^2(6+5x^4)}$

(d) $y^3 = \dfrac{1}{x^4(7+2x^2)}$

→ (e) $y^3 = \dfrac{1}{x^4(6-x^2)}$

The standard form is : $y' + \frac{2}{x}y = 4x^3y^4$.

We divide both sides by y^4: $y^{-4}y' + \frac{2}{x}y^{-3} = 4x^3$, $y \neq 0$

Substituting: $w = y^{-3}$ and $\frac{dw}{dx} = -3y^{-4}\frac{dy}{dx}$ in the equation above:

$$-\frac{1}{3}\frac{dw}{dx} + \frac{2}{x}w = 4x^3 \quad \text{or} \quad \frac{dw}{dx} - \frac{6}{x}w = -12x^3.$$

The integrating factor: $u(x) = e^{\int -\frac{6}{x}dx} = e^{-6\ln x} = x^{-6}$.

Hence, $x^{-6}\frac{dw}{dx} - 6x^{-7}w = -12x^{-3}$

or $\frac{d}{dx}\left(x^{-6}w\right) = -12x^{-3}$.

Integrating both sides: $x^{-6}w = 6x^{-2} + C$

Thus: $w = 6x^4 + Cx^6$. Substituting $w = y^{-3}$ leads to:

$$y^{-3} = x^4(6 + Cx^2)$$

Therefore: $$y^3 = \frac{1}{x^4(6+Cx^2)}$$

Q14. A particular solution $y_p(x)$ of the differential equation

$$2y'' + 4y' + 4y = e^{-x}\sec x$$

could be

(a) $\frac{e^{-x}}{2}(x\cos x + x\sin x \ln|\sin x|)$

(b) $e^{-x}(\cos x \ln|\cos x| + x\sin x)$

→ (c) $\frac{e^{-x}}{2}(\cos x \ln|\cos x| + x\sin x)$

(d) $\cos x \ln|\cos x| + x\sin x$

(e) $\frac{e^{-x}}{2}(\cos x \ln|\sin x| + x\sin x)$

Auxiliary equation of the homogeneous equation is:

$m^2 + 2m + 2 = 0$ ie $m = -1 \pm i$

$Y_c = e^{-x}[C_1\cos x + C_2 \sin x]$

The standard form of the equation: $y'' + 2y' + 2y = \frac{1}{2}e^{-x}\sec x$

$$W = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ e^{-x}(-\cos x - \sin x) & e^{-x}(\cos x - \sin x) \end{vmatrix} = e^{-2x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x}\sin x \\ \frac{1}{2}e^{-x}\sec x & e^{-x}(\cos x - \sin x) \end{vmatrix} = \frac{1}{2}e^{-2x}\tan x$$

$$W_2 = \begin{vmatrix} e^{-x}\cos x & 0 \\ e^{-x}(-\cos x - \sin x) & \frac{1}{2}e^{-x}\sec x \end{vmatrix} = \frac{1}{2}e^{-2x}$$

$u_1' = -\frac{1}{2}\tan x \Rightarrow u_1 = \frac{1}{2}\ln|\cos x|$

$u_2' = \frac{1}{2} \Rightarrow u_2 = \frac{1}{2}x$

Hence: $Y_p = \left[\frac{1}{2}\cos x \ln|\cos x| + \frac{1}{2}x\sin x\right]e^{-x}$

$= \frac{e^{-x}}{2}\left[\cos x \ln|\cos x| + x\sin x\right]$