King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

EXAM I – MATH 202 (Term 123)

June 25, 2013

Duration: **120** Minutes

Name: ______________________ **ID#:** ______________

Section/Instructor: ______________________ **Serial #:** ____________

- Provide **all necessary steps** with **clear writing**.
- **Mobiles** and **calculators** are **NOT allowed** in this exam.

Question #	Marks	Maximum Marks
Q1		13
Q2		12
Q3		16
Q4		17
Q5		12
Q6		17
Q7		13
Total		**100**

Q1. For the differential equation $y'' - 2y' + 2y = x + 1$,

(a) Verify that $y = 1 + \frac{x}{2} + C_1 e^x \cos x + C_2 e^x \sin x$ is a 2−parameter family of solutions.

$\cdot)\ y' = \frac{1}{2} + C_1(e^x\cos x - e^x\sin x) + C_2(e^x\sin x + e^x\cos x)$

$= \frac{1}{2} + (C_1+C_2)e^x\cos x + (C_2 - C_1)e^x\sin x.$

$\cdot)\ y'' = (C_1+C_2)(e^x\cos x - e^x\sin x) + (C_2 - C_1)(e^x\sin x + e^x\cos x)$

$= 2C_2 e^x\cos x - 2C_1 e^x\sin x$

Thus:

$y'' - 2y' + 2y = 2C_2e^x\cos x - 2C_1e^x\sin x - 2\left[\frac{1}{2} + (C_1+C_2)e^x\cos x + (C_2-C_1)e^x\sin x\right] + 2\left(1 + \frac{x}{2} + C_1e^x\cos x + C_2e^x\sin x\right) = x+1.$

Therefore $y = 1 + \frac{x}{2} + C_1e^x\cos x + C_2e^x\sin x$ is a 2-parameter family of solutions

(b) Find a solution of the initial value problem consisting of the above differential equation and the initial conditions $y(0) = 3$, $y'(0) = \frac{3}{2}$.

$y(0) = 1 + C_1 = 3 \Leftrightarrow C_1 = 2$

$y'(0) = \frac{1}{2} + C_1 + C_2 = \frac{3}{2} \Leftrightarrow C_2 = -1$

Hence the solution of this initial value problem is given by:

$y = 1 + \frac{x}{2} + 2e^x\cos x - e^x\sin x$

Q2. Solve the initial value problem

$$\frac{1}{x}y' = e^{x-y} + e^{-y}, \ y(1) = -\ln 2.$$

The equation can be written as:

$$e^{y}dy = x(e^{x}+1)\,dx$$

Now by integrating both sides, we have:

$\int e^{y}dy = \int x(e^{x}+1)dx$, which is equivalent to:

$$e^{y} = \frac{x^2}{2} + e^{x}(x-1) + C.$$

Since $y(1) = -\ln 2$, we have:

$$e^{-\ln 2} = \frac{1}{2} + C \quad \underline{i.e} \quad C = 0$$

Therefore the solution of this initial value problem is:

$$e^{y} = \frac{x^2}{2} + e^{x}(x-1).$$

Q3. (a) Show that $(x+1)y' + 2y = e^x$ is not exact.

$(x+1)\frac{dy}{dx} + 2y = e^x \Rightarrow (2y - e^x)dx + (x+1)dy = 0$

Let $M(x,y) = 2y - e^x$ and $N(x,y) = (x+1)$.

Since $\frac{\partial M(x,y)}{\partial y} = 2$ and $\frac{\partial N(x,y)}{\partial x} = 1$ are not equal, the equation is not exact

(b) Find an integrating factor μ that makes the equation in (a) exact.

It is easy to see that: $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-1}{x+1} = \frac{1}{x+1}$ is a function of x only.

So $\mu = e^{\int \frac{dx}{x+1}} = e^{\ln|x+1|} = x+1$ is an integrating factor that will make the equation exact

(c) Use (b) to solve the equation in (a).

We multiply both sides of the equation by μ, and we obtain.

$(x+1)(2y - e^x)dx + (x+1)^2 dy = 0$ Since this equation is exact, there exists a function $f(x,y)$ such that:

$\frac{\partial f}{\partial y} = (x+1)^2 \Rightarrow f(x,y) = y(x+1)^2 + g(x)$

and $\frac{\partial f}{\partial x} = (x+1)(2y - e^x)$.

Now, we differentiate f with respect to x, we obtain:

$\frac{\partial f}{\partial x} = 2y(x+1) + g'(x) = (x+1)(2y - e^x)$. Thus

$g'(x) = -(x+1)e^x$ i.e $g(x) = -xe^x$ Hence,

$f(x,y) = y(x+1)^2 - xe^x$. Therefore, the implicit form of the solution is: $y(x+1)^2 - xe^x = c$, where c is a constant.

Q4. (a) Solve the initial value problem

$$(1-x)y' + xy = e^x \ln(x+1), \; y(0) = 2.$$

The standard form of the equation is:

$$y' + \frac{x}{1-x} y = \frac{e^x}{1-x} \ln(x+1).$$

An integrating factor is given by:

$$e^{\int \frac{x}{1-x} dx} = e^{-\int (1 - \frac{1}{1-x}) dx} = e^{-x + \ln|1-x|} = (1-x)e^{-x}$$

Multiplying both sides of the standard form by the integrating factor implies: $\frac{d}{dx}\left((1-x)e^{-x} y\right) = \ln(x+1).$

Thus $(1-x)e^{-x} y = \int \ln(x+1) dx = (x+1)\ln(x+1) - x + C.$

Hence $y = \frac{e^x}{1-x}\left[(x+1)\ln(x+1) - x + C\right].$

Since $y(0) = C = 2$, we have that the solution of this initial value problem is given by:

$$y = \frac{e^x}{1-x}\left[(x+1)\ln(x+1) - x + 2\right]$$

(b) Give the largest interval in which the solution is defined.

$I = (-1, 1)$

Q5. Find all possible values of m for which $y = x^m$ is a solution of the differential equation

$$2x^2y'' - 3xy' - 3y = 0,$$

(a) on $(-\infty, +\infty)$,

(b) on $(0, +\infty)$.

Substituting $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$ in the equation implies:

$$2x^2 m(m-1)x^{m-2} - 3xmx^{m-1} - 3x^m = 0$$

Thus $2m(m-1) - 3m - 3 = 0$

or $2m^2 - 5m - 3 = 0$

or $(2m+1)(m-3) = 0$.

Hence $m = -\frac{1}{2}$ or 3

(a) On $(-\infty, +\infty)$, the only possible value of m is 3.

(b) on $(0, +\infty)$, the possible values of m are 3 and $-\frac{1}{2}$.

Q6. (a) Use a suitable substitution to reduce the differential equation

$$xy^2\frac{dy}{dx} + y^3 = x^3$$

to a separable equation. Do not solve the new equation.

The equation can be written as:

$$(y^3 - x^3)\,dx + xy^2\,dy = 0,$$ which is homogeneous.

We let $y = ux$, So that $dy = u\,dx + x\,du$. Then the homogeneous equation becomes:

$$(u^3x^3 - x^3)\,dx + u^2x^3(u\,dx + x\,du) = 0$$

or $$x^3(2u^3 - 1)\,dx + u^2x^4\,du = 0$$

or $$\frac{dx}{x} + \frac{u^2}{2u^3 - 1}\,du = 0$$ which is a separable equation.

(b) Use an appropriate substitution to transform the differential equation

$$y^{\frac{5}{2}}\frac{dy}{dx} - 2xy^{\frac{7}{2}} = y\sin x$$

to a linear equation. Do not solve the new equation.

The equation can be written as $\frac{dy}{dx} - 2xy = y^{-3/2}\sin x$, which is a Bernoulli's equation with $n = -3/2$. Now, we let $u = y^{1+3/2} = y^{5/2}$ So that $y = u^{2/5}$. Thus $\frac{dy}{dx} = \frac{2}{5}u^{-3/5}\frac{du}{dx}$, and the Bernoulli's equation becomes:

$$\frac{2}{5}u^{-3/5}\frac{du}{dx} - 2xu^{2/5} = u^{-3/5}\sin x \quad \text{i.e.} \quad \frac{du}{dx} - 2xu = \sin x$$

which is linear in u.

Q7. Initially 80 milligrams of a radioactive substance was present. After 7 hours the mass had decreased by 75%. If the rate of decay is proportional to the amount of the substance present at time t, find the half-life of the substance.

$$\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k\,dt \Rightarrow \int \frac{dP}{P} = \int k\,dt$$
$$\Rightarrow \ln|P| = kt + C_1$$
$$\Rightarrow P = Ce^{kt}, \text{ with } C = e^{C_1}$$

Since $P(0) = C = 80$, we have that $P = 80e^{kt}$

$P(7) = 80e^{7k} = 80 - \left(\frac{75}{100} \times 80\right) = 80 - 60 = 20.$

Thus $e^{7k} = \frac{1}{4}$ $\underline{\text{i.e}}$ $7k = \ln\left(\frac{1}{4}\right)$ $\underline{\text{i.e}}$ $k = -\frac{2}{7}\ln 2$

Hence $P(t) = 80\, e^{-\frac{2}{7}t\ln 2}$

Let T be the half-time of the substance Then:

$$80\,e^{-\frac{2}{7}T\ln 2} = 40, \text{ and so } e^{-\frac{2}{7}T\ln 2} = \frac{1}{2}$$

$\underline{\text{Thus}}$: $-\frac{2}{7}T\ln 2 = \ln\left(\frac{1}{2}\right) = -\ln 2$

Hence $T = 3.5$ hours.