KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 102 TERM 123
MAJOR EXAM II

KEY

Exercise # 1: (8 pts) Find

$$I = \int \mathrm{e}^{2x} \sin(\mathrm{e}^{x}) dx$$

Ans: Let $t = \mathrm{e}^x$. $dt = \mathrm{e}^x dx$. So, $I = \int t \sin t dt$.

Integration by parts with $u = t$ and $dv = \sin t dt$, gives, $du = dt$ and $v = -\cos t$ so that

$$\begin{aligned} I &= uv - \int v du = -t\cos t + \int \cos t dt \\ &= -t\cos t + \sin t + C \\ &= -\mathrm{e}^x \cos \mathrm{e}^x + \sin \mathrm{e}^x + C \end{aligned}$$

where C is an arbitrary constant.

Exercise # 2: (8 pts) Find

$$I = \int \sec^4(5x) dx$$

Ans:

$$\begin{aligned} I &= \int \sec^2(5x)[1 + \tan^2(5x)]dx \\ &= \int \sec^2(5x)dx + \int \sec^2(5x)\tan^2(5x)dx \\ &= \frac{1}{5}\tan(5x) + \frac{1}{15}\tan^3(5x) + C \end{aligned}$$

where C is an arbitrary constant.

Exercise #3: (8 pts) Evaluate the following integral

$$I = \int_1^3 \frac{dx}{x^2 - x + 2}$$

Ans:

$$\begin{aligned} I &= \int_1^3 \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \\ &= \frac{1}{\frac{\sqrt{7}}{2}} \tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)_{|x=3} - \frac{1}{\frac{\sqrt{7}}{2}} \tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)_{|x=1} \\ &= \frac{2}{\sqrt{7}}\left(\tan^{-1}\frac{5}{\sqrt{7}} - \tan^{-1}\frac{1}{\sqrt{7}}\right) \end{aligned}$$

or if we use the identity

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

we get,

$$\begin{aligned} I &= \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{\frac{5}{\sqrt{7}} - \frac{1}{\sqrt{7}}}{1 + \frac{5}{\sqrt{7}}\frac{1}{\sqrt{7}}}\right) = \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{4}{\sqrt{7}}\frac{7}{12}\right) \\ &= \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{\sqrt{7}}{3}\right) \end{aligned}$$

Exercise #4: (9 pts) Find

$$I = \int \frac{x^3 - 4x - 1}{(x^2-1)(x-1)} dx$$

Ans: The partial fractions decomposition of the integrand is

$$\begin{aligned} \frac{x^3 - 4x - 1}{(x^2-1)(x-1)} &= \frac{x^3 - 4x - 1}{x^3 - x^2 - x + 1} = 1 + \frac{x^2 - 3x - 2}{x^3 - x^2 - x + 1} \\ &= 1 + \frac{A}{x+1} + \frac{B}{x-1} + \frac{D}{(x-1)^2} \end{aligned}$$

where $A=\frac{1}{2}, B=\frac{1}{2}, D=-2$. Thus,

$$\frac{x^3-4x-1}{(x^2-1)(x-1)} = 1+\frac{1/2}{x+1}+\frac{1/2}{x-1}-\frac{2}{(x-1)^2}$$

leading to

$$I = x+\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+\frac{2}{x-1}+C$$

where C is an arbitrary constant or equivalently,

$$I = x+\frac{1}{2}\ln|x^2-1|+\frac{2}{x-1}+C$$

Exercise # 5: (8 pts) Evaluate the improper integral

$$I=\int_0^\infty \frac{x^2}{(x^3+1)^2}dx$$

Ans: This is an improper integral, so,

$$\begin{aligned} I &= \lim_{t\to\infty}\int_0^t \frac{x^2}{(x^3+1)^2}dx \\ &= \frac{1}{3}\lim_{t\to\infty}\left\{-\frac{1}{x^3+1}\right\}_{x=0}^{t} \\ &= \frac{1}{3}\lim_{t\to\infty}\left\{-\frac{1}{t^3+1}+1\right\} \\ &= \frac{1}{3} \end{aligned}$$

Exercise # 6: (9 pts) Evaluate the improper integral

$$I=\int_0^2 \frac{dx}{(2x-1)^{2/3}}$$

Ans: This is an improper integral, so, with a singularity at $\frac{1}{2}$,we have

$$\begin{aligned} I_1 &= \lim_{t\to(\frac{1}{2})_-}\int_0^t \frac{dx}{(2x-1)^{2/3}} = \lim_{t\to(\frac{1}{2})_-}\left[\frac{3}{2}(2x-1)^{\frac{1}{3}}\right]_{x=0}^{t} \\ &= \lim_{t\to(\frac{1}{2})_-}\left[\frac{3}{2}(2t-1)^{\frac{1}{3}}-\frac{3}{2}(-1)^{\frac{1}{3}}\right] = \frac{3}{2} \end{aligned}$$

and

$$
\begin{aligned}
I_2 &= \lim_{t\to\left(\frac{1}{2}\right)_+} \int_t^2 \frac{dx}{(2x-1)^{2/3}} = \lim_{t\to\left(\frac{1}{2}\right)_+} \left[\frac{3}{2}(2x-1)^{\frac{1}{3}}\right]_{x=t}^2 \\
&= \lim_{t\to\left(\frac{1}{2}\right)_+} \left[\frac{3}{2}(4-1)^{\frac{1}{3}} - \frac{3}{2}(2t-1)^{\frac{1}{3}}\right] = \frac{3}{2}3^{\frac{1}{3}}
\end{aligned}
$$

Thus,

$$
I = I_1 + I_2 = \frac{3}{2} + \frac{3}{2}3^{\frac{1}{3}} = \frac{3}{2}\left(1 + 3^{\frac{1}{3}}\right)
$$

Exercise # 7: (9 pts) Evaluate

$$
I = \int \sin^3 x \cos^2 x dx
$$

Ans: We have

$$
\begin{aligned}
I &= \int \sin^2 x \cos^2 x \sin x dx = \int \left(1 - \cos^2 x\right) \cos^2 x \sin x dx \\
&= \int \cos^2 x \sin x dx - \int \cos^4 x \sin x dx \\
&= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C
\end{aligned}
$$

where C is an arbitrary constant.

Exercise #8: (8 pts) Show that

$$
\int \operatorname{sech}(x) dx = \tan^{-1}(\sinh x) + C
$$

where C is an arbitrary constant.

Ans: Since,

$$
\left(\tan^{-1}(\sinh x)\right)' = \frac{\cosh x}{1 + \sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = \operatorname{sech} x
$$

we have,

$$
\int \operatorname{sech}(x) dx = \tan^{-1}(\sinh x) + C
$$

where C is an arbitrary constant.

Exercise #9: (8 pts) Find the derivative of

$$y = \sinh^{-1}(x^2 + x)$$

Ans: We have

$$\frac{dy}{dx} = \frac{2x+1}{\sqrt{1+(x^2+x)^2}}$$

Exercise # 10: (8 pts) Find

$$I = \int \frac{7^{1/x}}{x^2} dx$$

Ans: We have

$$I = \int \frac{e^{\frac{\ln 7}{x}}}{x^2} dx$$

Taking $v = \frac{1}{x}$, $dv = -\frac{1}{x^2} dx$ and the integral becomes,

$$\begin{aligned} I &= -\int e^{v \ln 7} dv = -\frac{1}{\ln 7} e^{v \ln 7} + C \\ &= -\frac{1}{\ln 7} 7^{1/x} + C \end{aligned}$$

where C is an arbitrary constant.

Exercisex # 11: (8 pts) evaluate the following integral

$$I = \int \frac{3x^2+1}{x^2-2x+1} dx$$

Ans: We have,

$$\begin{aligned} \frac{3x^2+1}{x^2-2x+1} &= 3 + \frac{6x-2}{x^2-2x+1} \\ &= 3 + \frac{6}{x-1} + \frac{4}{(x-1)^2} \end{aligned}$$

so that,

$$I = 3x + 6\ln|x-1| - \frac{4}{x-1} + C$$

where C is an arbitrary constant.

Exercise # 12: (9 pts) Find

$$I = \int \frac{dx}{(4x^2+9)^2}$$

Ans: Let $\frac{2x}{3} = \tan\theta$ thus $x = \frac{3}{2}\tan\theta$ and $dx = \frac{3}{2}\sec^2\theta d\theta$

$$\begin{aligned} I &= \int \frac{\frac{3}{2}\sec^2\theta d\theta}{\left(9\tan^2\theta + 9\right)^2} \\ &= \frac{3}{2}\int \frac{\sec^2\theta}{\left(9\sec^2\theta\right)^2} = \frac{1}{54}\int \frac{1}{\sec^2\theta} d\theta \\ &= \frac{1}{54}\int \cos^2\theta d\theta = \frac{1}{108}\left[\theta + \sin\theta\cos\theta\right] + C \end{aligned}$$

But,

$$\theta = \tan^{-1}\left(\frac{2x}{3}\right) \ , \ \sin\theta = \frac{2x}{\sqrt{4x^2+9}} \ , \ \cos\theta = \frac{3}{\sqrt{4x^2+9}}$$

so,

$$\begin{aligned} I &= \frac{1}{108}\left[\tan^{-1}\left(\frac{2x}{3}\right) + \frac{2x}{\sqrt{4x^2+9}}\frac{3}{\sqrt{4x^2+9}}\right] + C \\ &= \frac{1}{108}\tan^{-1}\left(\frac{2x}{3}\right) + \frac{1}{18}\frac{x}{4x^2+9} + C \end{aligned}$$

where C is an arbitrary constant.