

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS  
 DEPARTMENT OF MATHEMATICS AND STATISTICS  
 MATH 102 TERM 123  
**MAJOR EXAM II**

# KEY

**Exercise # 1:** (8 pts) Find

$$I = \int e^{2x} \sin(e^x) dx$$

**Ans:** Let  $t = e^x$ .  $dt = e^x dx$ . So,  $I = \int t \sin t dt$ .

Integration by parts with  $u = t$  and  $dv = \sin t dt$ , gives,  $du = dt$  and  $v = -\cos t$  so that

$$\begin{aligned} I &= uv - \int v du = -t \cos t + \int \cos t dt \\ &= -t \cos t + \sin t + C \\ &= -e^x \cos e^x + \sin e^x + C \end{aligned}$$

where  $C$  is an arbitrary constant.

**Exercise # 2:** (8 pts) Find

$$I = \int \sec^4(5x) dx$$

**Ans:**

$$\begin{aligned} I &= \int \sec^2(5x)[1 + \tan^2(5x)] dx \\ &= \int \sec^2(5x) dx + \int \sec^2(5x) \tan^2(5x) dx \\ &= \frac{1}{5} \tan(5x) + \frac{1}{15} \tan^3(5x) + C \end{aligned}$$

where  $C$  is an arbitrary constant.

**Exercise #3:** (8 pts) Evaluate the following integral

$$I = \int_1^3 \frac{dx}{x^2 - x + 2}$$

**Ans:**

$$\begin{aligned} I &= \int_1^3 \frac{dx}{(x - \frac{1}{2})^2 + (\frac{\sqrt{7}}{2})^2} \\ &= \frac{1}{\frac{\sqrt{7}}{2}} \tan^{-1} \left( \frac{x - \frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) \Big|_{x=3} - \frac{1}{\frac{\sqrt{7}}{2}} \tan^{-1} \left( \frac{x - \frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) \Big|_{x=1} \\ &= \frac{2}{\sqrt{7}} \left( \tan^{-1} \frac{5}{\sqrt{7}} - \tan^{-1} \frac{1}{\sqrt{7}} \right) \end{aligned}$$

or if we use the identity

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

we get,

$$\begin{aligned} I &= \frac{2}{\sqrt{7}} \tan^{-1} \left( \frac{\frac{5}{\sqrt{7}} - \frac{1}{\sqrt{7}}}{1 + \frac{5}{\sqrt{7}} \cdot \frac{1}{\sqrt{7}}} \right) = \frac{2}{\sqrt{7}} \tan^{-1} \left( \frac{4}{\sqrt{7}} \frac{7}{12} \right) \\ &= \frac{2}{\sqrt{7}} \tan^{-1} \left( \frac{\sqrt{7}}{3} \right) \end{aligned}$$

**Exercise #4:** (9 pts) Find

$$I = \int \frac{x^3 - 4x - 1}{(x^2 - 1)(x - 1)} dx$$

**Ans:** The partial fractions decomposition of the integrand is

$$\begin{aligned} \frac{x^3 - 4x - 1}{(x^2 - 1)(x - 1)} &= \frac{x^3 - 4x - 1}{x^3 - x^2 - x + 1} = 1 + \frac{x^2 - 3x - 2}{x^3 - x^2 - x + 1} \\ &= 1 + \frac{A}{x+1} + \frac{B}{x-1} + \frac{D}{(x-1)^2} \end{aligned}$$

where  $A = \frac{1}{2}$ ,  $B = \frac{1}{2}$ ,  $D = -2$ . Thus,

$$\frac{x^3 - 4x - 1}{(x^2 - 1)(x - 1)} = 1 + \frac{1/2}{x+1} + \frac{1/2}{x-1} - \frac{2}{(x-1)^2}$$

leading to

$$I = x + \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + \frac{2}{x-1} + C$$

where  $C$  is an arbitrary constant or equivalently,

$$I = x + \frac{1}{2} \ln|x^2 - 1| + \frac{2}{x-1} + C$$

**Exercise # 5:** (8 pts) Evaluate the improper integral

$$I = \int_0^\infty \frac{x^2}{(x^3 + 1)^2} dx$$

**Ans:** This is an improper integral, so,

$$\begin{aligned} I &= \lim_{t \rightarrow \infty} \int_0^t \frac{x^2}{(x^3 + 1)^2} dx \\ &= \frac{1}{3} \lim_{t \rightarrow \infty} \left\{ -\frac{1}{x^3 + 1} \right\}_{x=0}^t \\ &= \frac{1}{3} \lim_{t \rightarrow \infty} \left\{ -\frac{1}{t^3 + 1} + 1 \right\} \\ &= \frac{1}{3} \end{aligned}$$

**Exercise # 6:** (9 pts) Evaluate the improper integral

$$I = \int_0^2 \frac{dx}{(2x - 1)^{2/3}}$$

**Ans:** This is an improper integral, so, with a singularity at  $\frac{1}{2}$ , we have

$$\begin{aligned} I_1 &= \lim_{t \rightarrow (\frac{1}{2})_-} \int_0^t \frac{dx}{(2x - 1)^{2/3}} = \lim_{t \rightarrow (\frac{1}{2})_-} \left[ \frac{3}{2} (2x - 1)^{\frac{1}{3}} \right]_{x=0}^t \\ &= \lim_{t \rightarrow (\frac{1}{2})_-} \left[ \frac{3}{2} (2t - 1)^{\frac{1}{3}} - \frac{3}{2} (-1)^{\frac{1}{3}} \right] = \frac{3}{2} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \lim_{t \rightarrow (\frac{1}{2})^+} \int_t^2 \frac{dx}{(2x-1)^{2/3}} = \lim_{t \rightarrow (\frac{1}{2})^+} \left[ \frac{3}{2}(2x-1)^{\frac{1}{3}} \right]_{x=t}^2 \\ &= \lim_{t \rightarrow (\frac{1}{2})^+} \left[ \frac{3}{2}(4-1)^{\frac{1}{3}} - \frac{3}{2}(2t-1)^{\frac{1}{3}} \right] = \frac{3}{2}3^{\frac{1}{3}} \end{aligned}$$

Thus,

$$I = I_1 + I_2 = \frac{3}{2} + \frac{3}{2}3^{\frac{1}{3}} = \frac{3}{2} \left( 1 + 3^{\frac{1}{3}} \right)$$

**Exercise # 7: (9 pts)** Evaluate

$$I = \int \sin^3 x \cos^2 x dx$$

**Ans:** We have

$$\begin{aligned} I &= \int \sin^2 x \cos^2 x \sin x dx = \int (1 - \cos^2 x) \cos^2 x \sin x dx \\ &= \int \cos^2 x \sin x dx - \int \cos^4 x \sin x dx \\ &= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C \end{aligned}$$

where  $C$  is an arbitrary constant.

**Exercise #8: (8 pts)** Show that

$$\int \operatorname{sech}(x) dx = \tan^{-1}(\sinh x) + C$$

where  $C$  is an arbitrary constant.

**Ans:** Since,

$$(\tan^{-1}(\sinh x))' = \frac{\cosh x}{1 + \sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = \operatorname{sech} x$$

we have,

$$\int \operatorname{sech}(x) dx = \tan^{-1}(\sinh x) + C$$

where  $C$  is an arbitrary constant.

**Exercise #9:** (8 pts) Find the derivative of

$$y = \sinh^{-1}(x^2 + x)$$

**Ans:** We have

$$\frac{dy}{dx} = \frac{2x+1}{\sqrt{1+(x^2+x)^2}}$$

**Exercise # 10:** (8 pts) Find

$$I = \int \frac{7^{1/x}}{x^2} dx$$

**Ans:** We have

$$I = \int \frac{e^{\frac{\ln 7}{x}}}{x^2} dx$$

Taking  $v = \frac{1}{x}$ ,  $dv = -\frac{1}{x^2} dx$  and the integral becomes,

$$\begin{aligned} I &= - \int e^{v \ln 7} dv = -\frac{1}{\ln 7} e^{v \ln 7} + C \\ &= -\frac{1}{\ln 7} 7^{1/x} + C \end{aligned}$$

where  $C$  is an arbitrary constant.

**Exercisex # 11:** (8 pts) evaluate the following integral

$$I = \int \frac{3x^2 + 1}{x^2 - 2x + 1} dx$$

**Ans:** We have,

$$\begin{aligned} \frac{3x^2 + 1}{x^2 - 2x + 1} &= 3 + \frac{6x - 2}{x^2 - 2x + 1} \\ &= 3 + \frac{6}{x-1} + \frac{4}{(x-1)^2} \end{aligned}$$

so that,

$$I = 3x + 6 \ln|x - 1| - \frac{4}{x - 1} + C$$

where  $C$  is an arbitrary constant.

**Exercise # 12:** (9 pts) Find

$$I = \int \frac{dx}{(4x^2 + 9)^2}$$

**Ans:** Let  $\frac{2x}{3} = \tan \theta$  thus  $x = \frac{3}{2} \tan \theta$  and  $dx = \frac{3}{2} \sec^2 \theta d\theta$

$$\begin{aligned} I &= \int \frac{\frac{3}{2} \sec^2 \theta d\theta}{(9 \tan^2 \theta + 9)^2} \\ &= \frac{3}{2} \int \frac{\sec^2 \theta}{(9 \sec^2 \theta)^2} = \frac{1}{54} \int \frac{1}{\sec^2 \theta} d\theta \\ &= \frac{1}{54} \int \cos^2 \theta d\theta = \frac{1}{108} [\theta + \sin \theta \cos \theta] + C \end{aligned}$$

But,

$$\theta = \tan^{-1} \left( \frac{2x}{3} \right), \quad \sin \theta = \frac{2x}{\sqrt{4x^2 + 9}}, \quad \cos \theta = \frac{3}{\sqrt{4x^2 + 9}}$$

so,

$$\begin{aligned} I &= \frac{1}{108} \left[ \tan^{-1} \left( \frac{2x}{3} \right) + \frac{2x}{\sqrt{4x^2 + 9}} \frac{3}{\sqrt{4x^2 + 9}} \right] + C \\ &= \frac{1}{108} \tan^{-1} \left( \frac{2x}{3} \right) + \frac{1}{18} \frac{x}{4x^2 + 9} + C \end{aligned}$$

where  $C$  is an arbitrary constant.