HW 2

Exercise 1: If ϕ is a characteristic function, show that $Re\{1-\phi(t)\} \geq \frac{1}{4} Re\{1-\phi(2t)\}$ and deduce that:

$$1-|\phi(2t)| \leq 8(1-|\phi(t)|)$$

Exercise 2: Let $X_1, X_2, \ldots, X_n$ be independent variables, X_i being $N(\mu_i, 1)$ and let $Y = X_1^2 + X_2^2 + \cdots + X_n^2$.
Show that the characteristic function of Y is:

$$\phi_Y(t) = \frac{1}{(1-2it)^{n/2}} \exp\left(\frac{it\theta}{1-2it}\right)$$

where $\theta = \mu_1^2 + \mu_2^2 + \cdots + \mu_n^2$

Exercise 3: If X and Y are continuous random variables, show that

$$\int_{-\infty}^{\infty} \phi_X(y) f_Y(y) e^{-ity} \, dy = \int_{-\infty}^{\infty} \phi_Y(x-t) f_X(x) \, dx.$$

Exercise 4: Show that the exponential distribution is the only probability distribution satisfying the Lack of memory property.
(Lack of memory property: $P(\eta > t+s \mid \eta > s) = P(\eta > t)$.

Exercise 5: Given that for any fixed $t \geq 0$; $N^t(s) = N(t+s) - N(t)$, $s \geq 0$ is a Poisson process independent of $N(t)$ with Same probability law as $N(s)$.
Show that for any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ the increments:
$N(t_1), N(t_2)-N(t_1), N(t_3)-N(t_2), \ldots, N(t_n)-N(t_{n-1})$ are independent and have the same probability distribution as:

$$N(t_1), N(t_2-t_1), N(t_3-t_2), \ldots, N(t_n-t_{n-1}).$$