King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

<u>Course:</u> ***Introduction to stochastic differential equations and applications to Mathematical Finance***
Math 590 @ Math 690

Final Exam – 2012–2013 (122)
Sunday, May 19, 2013

Allowed Time: 150 minutes

Instructor: **Dr. Boubaker Smii**

Name: ______________________________

ID #: ____________________

Section #: ___________ Serial Number: ___________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. <u>No points for answers without justification !</u>

Question #	Grade	Maximum Points
1		**13**
2		**14**
3		**19**
4		**11**
5		**23**
Total:		**80**

Exercise 1:

1- Let X be a random variable exponentially distributed with parameter λ.
Prove that $\mathbb{E}(e^{itX}) = \frac{\lambda}{(\lambda - i\,t)}, \ t \geq 0$.

2- Assume that X is Poisson distributed with parameter λ.
Find the characteristic function of X.

Exercise 2:

Let B_t be a standard Brownian motion.

1- Write down the stochastic differential equation corresponding to the process $Y_t = B_t^4$ and deduce $\mathbb{E}(B_t^4)$.

2-Write down the stochastic differential equation corresponding to the process $Y_t = t\,B_t$.

3-Verify that $X_t = e^{B_t - \frac{t}{2}}$ satisfies the stochastic differential equation: $dX_t = X_t\,dB_t$.

Exercise 3:

Let $B_t \in \mathbb{R}$, $B_0 = 0$. Define $X_k(t) = \mathbb{E}(B_t^k)$, $k = 0, 1, 2, ...; t \geq 0$.

1)- Prove that $X_k = \frac{k(k-1)}{2} \int_0^t X_{k-2}(s)\, ds, \ \ k \geq 2.$

2)- Deduce $\mathbb{E}(B_t^4)$ and $\mathbb{E}(B_t^6)$

3)- Show that $\mathbb{E}(B_t^{2k+1}) = 0$ and $\mathbb{E}(B_t^{2k}) = \frac{(2k)!t^k}{2^k\, k!}; \ \ k = 1, 2, ...$

Exercise 4:

Assume that you exercise an option at a fixed price K and a maturity time T. The value V_t of your portfolio at time t is given by:

$$V_t = u(T-t,\, X_t), \;\; t \in [0,\, T],$$

for some smooth deterministic function $u(t,\, x)$ and X_t a stochastic process satisfying:

$$X_t = X_0 + c\int_0^t X_s\, ds + \sigma\int_0^t X_s\, dB_s, \quad c > 0,\, \sigma > 0.$$

1- Find the European call option.

2-Express V_t in terms of u_1, u_2 and u_{22}.

Exercise 5:

A- The Black-Scholes-Merton model for growth with uncertain rate of return, is the value of §1 after time t, invested in a saving account. It is described by the following stochastic differential equation:

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dB_t,\ \ \mu, \sigma > 0 \tag{a}$$

Prove that the solution of the SDE (a) is given by a Geometric Brownian motion.

B-Let S_t be the price of a stock at time t. Suppose that stock price is modelled as a geometric Brownian motion $S_t = S_0\, e^{\mu\, t + \sigma\, B_t}$, where B_t is a standard Brownian motion.

1- Suppose that the parameter values are $\mu = 0.055$ and $\sigma = 0.07$.
Given that $S_5 = 100$, find the probability that S_{10} is greater than 150. (you may express the result as $\Phi(\alpha)$, where Φ is the standard Normal distribution function and α a real number.)

2- Now assume that μ and σ are fixed parameters and the initial value of the stock is $S_0 = 1$.
i)- Find the median of S_t and the expectation of S_t.
ii)- Given that $\mu = -\frac{1}{2}\sigma^2$. State, with justifications, whether or not the stock would be a good long term investment in this case.