King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 590
Exam I– 2012–2013 (122)
Thursday, April 25, 2013

Allowed Time: 120 minutes

Instructor: **Dr. Boubaker Smii**

Name: ____

ID #: ______________________________

Section #: ________________ Serial Number: ________________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification !

Question #	**Grade**	**Maximum Points**
1		**13**
2		**07**
3		**16**
4		**12**
5		**12**
Total:		**60**

Exercise 1:

Consider the standard Brownian motion $\{B_t,\, t \geq 0\}$.

a)- Show whether or not $V_t = \sqrt{t}\, B_t$ is a standard Brownian motion.

b)- Find $\mathbb{E}(\mid B_t - B_s \mid^2)$.

c)- Given that $\displaystyle\int_{\mathbb{R}} e^{\frac{-(x - i\,\lambda\, t)^2}{2t}}\, dx = \sqrt{2\,\pi}$, compute the characteristic function of B_t.

d)- Deduce from c) $\mathbb{E}(B_t^4)$.

Exercise 2:

Consider the Brownian motion with drift coefficient μ and variance parameter σ, given by $Y_t = \mu\, t + \sigma\, B_t$, where B_t is a standard Brownian motion.
Let $X_t = e^{Y_t}$, $t \geq 0$. For $0 \leq u \leq s < t$, and Y_s given, compute $E(X_t \mid X_u)$.

<u>Exercise 3:</u> Let X_t, Y_t be Itô processes in $\mathbb{R}$.

1)- Prove that:

$$X_t\,Y_t = X_0\,Y_0 + \int_0^t Y_s\,dX_s + \int_0^t X_s\,dY_s + \int_0^t dX_s\,dY_s$$

2)- Let $F_t = \exp(-\,\alpha\,B_t + \frac{1}{2}\,\alpha^2\,t)$.

i)- Find dF_t.

ii)- Solve the stochastic differential equation: $dY_t = r\,dt + \alpha\,Y_t\,dB_t$. (**Hint**: Use Part 1)).

Exercise 4:

A vibrating string subject of a stochastic force is modeled by the following stochastic differential equation:

$$\begin{cases} dX_1(t) = X_2(t)\,dt + \alpha\,dB_1(t) \\ dX_2(t) = -X_1(t)\,dt + \beta\,dB_2(t) \end{cases} \tag{a}$$

where $(B_1(t), B_2(t))$ is a 2-dimensional Brownian motion and α, β are constants.
Solve the stochastic differential equation (a).

Exercise 5:

The mean-reverting Ornstein-Uhlenbeck process is the solution X_t of the stochastic differential equation

$$dX_t = (m - X_t)\,dt + \sigma\,dB_t \tag{1}$$

where m, σ are real constants, $B_t \in \mathbb{R}$.

a)- Solve the stochastic differential equation (1).

b)- Find $\mathbb{E}(X_t)$ and $Var(X_t)$.