King Fahd University of Petroleum and Minerals
deaprtment of Mathematics and Statistics
Math 571: Final Exam (Take Home)

Name:..ID:.....................

Please provide the MATLAB (or Mathematica or Maple) codes and their outputs (table/graphs) for Problems #2 and #3. A bonus will be given for solving correctly Problem #3 using all the cited methods.

Problem #1: Consider the initial value problem

$$\frac{dy}{dt} = f(y,t), y(t_0) = y_0, t \in [t_0, T].$$

In integral form we have

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau), \tau) d\tau$$

giving

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(y(\tau), \tau) d\tau$$

Let $h = t_{n+1} - t_n$ "small", this leads to the one-parameter class of methods, **θ-method**,

$$y_{n+1} = y_n + h\left[(1-\theta) f(y_n, t_n) + \theta f(y_{n+1}, t_{n+1})\right] \tag{1}$$

for $\theta \in [0, 1]$.
Note that, for $\theta = 0$ we recover the Euler's method and for for $\theta = 1$, we get the backward Euler method.

1. Determine the first two terms of the residual r_n for the θ-method for general f. Do this as follows:

   (i) write down the Taylor expansion of $y'(t_{n+1})$ about t_n and recall $f(y(t_{n+1}), t_{n+1}) = y'(t_{n+1})$

   (ii) note that the residual is obtained upon substitution of the exact solution into the method:

   $$y(t_{n+1}) = y(t_n) + h\left[(1-\theta) y'(t_n) + \theta y'(t_{n+1})\right] - r_n \tag{2}$$

   (Hint: compare the above relation to the Taylo expansion of $y(t_{n+1})$ about time t_n). What is the order of accuracy? How does it depend on θ?

2. Prove that the θ-method converges for sufficiently smooth f . Proced as follows:

   (i) first show that the error satisfies

   $$(1 - h\theta L) \|e_{n+1}\| \leq (1 + h(1-\theta)L) \|e_n\| + (\theta - \frac{1}{2}) h^2 C_1 + (\theta - \frac{1}{3}) \frac{h^3}{2} C_2$$

where L is the Lipschitz constant of and C_1 and C_2 are positive constants (Hint: subtract 2 from 1.

(ii) next, show by induction on n that

$$\|e_n\| \leq \left[\left(\frac{1+h(1-\theta)L}{1-h\theta L}\right)^n - 1\right]\left(\frac{\theta-1/2}{L}hC_1 + \frac{\theta-1/3}{L}h^2C_2\right)$$

(iii) finally, show that for $\theta \in [0,1]$, the bound

$$\|e_n\| \leq \left[\exp\left(\frac{hnL}{1-hL}\right) - 1\right]\left(\frac{\theta-1/2}{L}hC_1 + \frac{\theta-1/3}{L}h^2C_2\right)$$

holds and conclude the convergence in the limit $h \to 0$, for fixed T. How does the convergence depend on θ?

3. Determine the stability region S for $\theta = 1/2$ and $\theta = 1$. Describe these regions in words.

Problem #2: Solve the initial value problem

$$\begin{cases} \frac{dy}{dt} = -e^{-y-t}. \; t \in [0,5] \\ y(0) = \ln 2 \end{cases}$$

using the Runge-Kutta methods of order two and four. Then evaluate $y(5)$ in each case and compare with exact value.

Problem #3: Solve the initial value problem

$$\begin{cases} \frac{dy}{dt} = -e^{-y-t}. \ t \in [0,5] \\ y(0) = \ln 2 \end{cases}$$

using **one of the following methods:**
(i) the Adomian Decomposition Method
(ii) the Variation Iteration Method
(iii) the Integral equation method (Numerically solve an appropriate integral equation)

Then evaluate $y(5)$ and compare with exact value.