King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Abstract Algebra (Math 551), Semester 122
Final Exam

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Throughout, and unless otherwise explicitly mentioned, R is an associative ring with $1_R \neq 0_R$, ${}_R\mathbf{Mod}$ (resp. $\mathbf{Mod}_R$) is the category of left (resp. right) R-modules and $\mathbf{Ab}$ is the category of Abelian groups.

Part I. (60 points) Solve **three** of the following four questions:

Q1. A left R-module M is *semisimple* iff $M = \mathrm{Soc}(M)$, where

$$\mathrm{Soc}(M) = \sum_{S \in \mathcal{S}(M)} S \text{ and } S(M) = \{S \mid S \leq_R M \text{ is a simple } R\text{-submodule}\}.$$

1. Show that ${}_RM$ is semisimple if and only if every R-submodule of M is a direct summand.
2. Show that

$$\mathrm{Soc}(M) = \bigcap_{L \in \mathcal{E}(M)} L, \text{ where } \mathcal{E}(M) = \{L \mid L \leq_R M \text{ is an essential } R\text{-submodule}\}.$$

3. Compute $\mathrm{Soc}(\mathbb{Z}_{600})$.

Q2. The *Jacobson radical* of the ring R is defined as

$$\mathrm{J}(R) := \bigcap_{\mathfrak{m} \in \mathrm{Max}({}_RR)} \mathfrak{m}, \text{ where } \mathrm{Max}({}_RR) \text{ is the set of maximal left ideals of } R.$$

1. Show that

$$\begin{aligned} \mathrm{J}(R) &= \{r \in R \mid 1 - ar \text{ has a left inverse for all } a \in R\}; \\ &= \{r \in R \mid rS = 0 \text{ for some simple } R\text{-module } S\}. \end{aligned}$$

2. Show that if R is left Artinian, then $R/\mathrm{J}(R)$ is semisimple.
3. Compute $\mathrm{J}(\mathbb{Z}/12\mathbb{Z})$ and $\mathrm{J}(U_n(\mathbb{F}))$, where $U_n(\mathbb{F})$ is the ring of upper triangular $n \times n$ matrices with entries in the field $\mathbb{F}$.

Q3. A left R-module F is *flat* iff the functor $- \otimes_R F : \mathbf{Mod}_R \longrightarrow \mathbf{Ab}$ is exact.

1. Show that ${}_RF$ is flat if and only if the canonical map $\mu_I : I \otimes_R F \longrightarrow IF$ is injective for every finitely generated right ideal I of R.
2. Suppose that R has no non-zero zero-divisors and assume that every finitely generated right ideal of R is principal. Show that ${}_RF$ is flat if and only if ${}_RF$ is torsion free.
3. Show that if a left R-module F is flat, then $\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is an injective right R-module.

Q4. A left R-module P is *projective* if and only if $\mathrm{Hom}_R(P, -) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is exact.

1. Show that ${}_RP$ is projective if and only if ${}_RP$ has a dual basis, *i.e.* there exists a subset $\{(p_\lambda, f_\lambda)\} \subseteq P \times P^*$ such that every $p \in P$ can be written as $p = \sum f_\lambda(p)p_\lambda$ with $f_\lambda(p) \neq 0$ for only a finite number of $\lambda \in \Lambda$.
2. Show that the ring R is semisimple iff every left R-module is projective.
3. Give an example of a projective module P over a commutative ring such that P^* is not projective.

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Part II. (20 points) Consider the *Prüfer group*

$$\mathbb{Z}_{p^\infty} = \sum_{k \in \mathbb{N}} \mathbb{Z}(\frac{1}{p^k} + \mathbb{Z}) \subseteq \mathbb{Q}/\mathbb{Z},$$

where p is a prime positive integer.

1. Find all $\mathbb{Z}$-submodules of $\mathbb{Z}_{p^\infty}$ and show that they form a chain.
2. Show that $\mathbb{Z}_{p^\infty}$ contains a unique simple $\mathbb{Z}$-module.
3. Show that $\mathbb{Z}_{p^\infty}$ is Artinian but not Noetherian.
4. Show that $\mathbb{Z}_{p^\infty}$ is an injective $\mathbb{Z}$-module.
5. Show that there is an essential embedding $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^\infty}$.

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Part III. (20 points) State which of the following statements are true and which are false.

1. Every simple ring is semisimple.
2. For every ideal of R with $I \subseteq \mathrm{J}(R)$, we have $\mathrm{J}(R/I) = \mathrm{J}(R)/I$.
3. Every ideal of a commutative ring is an intersection of finitely many irreducible ideals of R.
4. $\mathbb{Q}$ has a maximal $\mathbb{Z}$-submodule.
5. $\mathbb{Z}[\sqrt{-5}]$ is a Noetherian ring.
6. If R is a non-unital Noetherian ring, then $R[x]$ is Noetherian.
7. Every Dedekind UFD is a PID.
8. $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in $\mathbf{Ab}$.
9. Every commutative primitive ring is a field.
10. Every left primitive ring is right primitive.

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