King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Abstract Algebra (Math 551), Semester 122
Mid Term Exam

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Throughout, and unless otherwise explicitly mentioned, R is an associative ring with $1_R \neq 0_R$ and ${}_R\mathbf{Mod}$ (resp. $\mathbf{Mod}_R$) is the category of left (resp. right) R-modules.

Part I. (60 points) Solve **three** of the following four questions:

Q1. A left R-module P is *projective* iff $\mathrm{Hom}_R(P,-) : {}_R\mathbf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathbf{Mod}$ is exact.

1. Show that ${}_RP$ is projective if and only if P is a direct summand of a free left R-module.
2. Show that every direct sum of projective left R-modules is projective.
3. Give an example of a projective left R-module which is not free. Justify your claims.

Q2. A left R-module E is *injective* iff $\mathrm{Hom}_R(-,E) : {}_R\mathbf{Mod} \longrightarrow {}_{\mathbb{Z}}\mathbf{Mod}$ is exact.

1. Show that ${}_RE$ is injective if and only if the canonical map $\mathrm{Hom}_R(-,E) : \mathrm{Hom}_R(R,E) \longrightarrow \mathrm{Hom}_R(I,E)$ is surjective for every ideal I of R;
2. If R is a Noetherian commutative ring and $\{E_\lambda\}_\Lambda$ is a (possibly infinite) class of injective R-modules, then $\bigoplus_\Lambda E_\lambda$ is injective.
3. Give an example of an injective R-module E with a submodule $M \leq E$ which is not injective. Justify your claims.

Q3. Let $\{M_\lambda\}_\Lambda$ be a class of left R-modules and N a left R-module.

1. Show that we have a canonical isomorphism of Abelian groups
$$\mathrm{Hom}_R(N, \prod_\Lambda M_\lambda) \simeq \prod_\Lambda \mathrm{Hom}_R(N, M_\lambda).$$
2. Show that we have a canonical isomorphism of Abelian groups
$$\mathrm{Hom}_R(\bigoplus_\Lambda M_\lambda, N) \simeq \prod_\Lambda \mathrm{Hom}_R(M_\lambda, N).$$
3. Give an example in which
$$\mathrm{Hom}_R(\bigoplus_\Lambda M_\lambda, N) \not\simeq \bigoplus_\Lambda \mathrm{Hom}_R(M_\lambda, N).$$

Q4. Consider the commutative diagram of R-modules

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

1. Show that if both inner squares are pullbacks, then the outer rectangle is a pullback.
2. Show that if the right square and the outer rectangle are pullbacks, then the left square is a pullback.
3. Give an example in which the left square and the outer rectangle are pullbacks but the right square is not a pullback.

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Part II. (20 points) Let M be an R-module. An R-submodule $L \leq_M R$ is said to be *essential* (*large*) in M iff for every non-zero R-submodule $0 \neq K \leq_R M$, we have $K \cap L \neq 0$.

Q1. Let $f : L \longrightarrow M$ be a monomorphism of R-modules. Show that the following are equivalent:

(a) $f(L) \leq_R M$ is an essential R-submodule.

(b) For every R-linear map $g : M \longrightarrow N$, the following holds

$$g \circ f \text{ is a monomorphism } \Rightarrow g \text{ is a monomorphism.}$$

(c) For every epimorphism $g : M \longrightarrow N$, the following holds

$$g \circ f \text{ is a monomorphism } \Rightarrow g \text{ is an isomorphism.}$$

Q2. Let $K \overset{h}{\hookrightarrow} L \overset{f}{\hookrightarrow} M$ be two monomorphisms of R-modules. Show that $h(K) \subseteq L$ and $f(L) \subseteq M$ are essential if and only if $(f \circ h)(K) \subseteq M$ is essential.

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Part III. (20 points) State which of the following statements is true and which are false.

1. If $\varphi : R \longrightarrow S$ is a morphism of rings and $P \in \mathrm{Spec}(S)$, then $\varphi^{-1}(P) \in \mathrm{Spec}(R)$.
2. If K is a field, then $K[x, y]$ is a PID.
3. Every injective left R-module is divisible.
4. Every non-zero left R-module contains a maximal R-submodule.
5. $\mathbb{Q}$ is a free Abelian group.
6. For all left R-module L, M and N we have
$$L \cap (M \oplus N) \simeq (L \cap M) \oplus (L \cap N).$$
7. Every Abelian group is a subgroup of a divisible Abelian group.
8. If M is a free left R-module and β is a basis for ${}_RM$ with n elements, then we have an isomorphism of rings $\mathrm{End}_R(M) \simeq \mathbb{M}_n(R)$.
9. All bases of a finitely generated free left R-module have the same number of elements.
10. The direct product of injective left R-modules is injective.

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