King Fahd University of Petroleum & Minerals ID #
Department of Mathematics & Statistics
MATH 550/ Final Exam/ 122 Name

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1. [30pts] Let T be a linear operator on a finite-dimensional complex inner product space V.

(a) Prove that if W is a T-invariant subspace of V then the orthogonal complement $W^{\perp}$ of W is T^{*}-invariant.

(b) Prove that if T is self-adjoint then $(Tx|x)$ is real for each x in V

(c) Prove that if $(Tx|x)$ is real for some x in V then $((T - T^{*})\,x|x) \in \mathbb{R}$. Is it true that if $(Tx|x)$ is real for each x in V then T is self-adjoint? Justify.

(d) State the spectral theorem, and use it to show that if T is normal then there exists a linear operator U on V such that $T = U^2$.

(e) Prove that if T is normal then $\|Tx\| = \|T^{*}x\|$ for each x in V. Is the converse true? Justify.

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2. [30pts] (a) Let V be a finite-dimensional real vector space and let f be a symmetric bilinear form on V. For any subspace W of V, let $W^{\perp} = \{v \in V : f\,(v, w) = 0\ \forall w \in W\}$

(i) Prove that $W^{\perp}$ is a subspace of V and that $W \subseteq W^{\perp\perp}$

(ii) Prove that $V = \{0\}^{\perp}$ and also prove that $V^{\perp} = \{0\}$ iff f is non-degenerate

(iii) Show that $rank(f) = \dim V - \dim V^{\perp}$

(b) Compute the rank and signature of the bilinear form g on $\mathbb{R}^3$ given by $g\left((x, y, z), (x', y', z')\right) = xy' + x'y$.

(c) Let $h : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the skew-symmetric bilinear form with matrix representation $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (in the standard ordered basis of $\mathbb{R}^2$). Prove that $\langle (1, 1) \rangle = \{(x, y) \in \mathbb{R}^2 : h\left((x, y), (1, 1)\right) = 0\}$. Is this still true if we replace $(1, 1)$ by any vector (a, b) of $\mathbb{R}^2$? Justify.

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3. [20pts] (a) A complex matrix A has distinct eigenvalues a, b, c with respective multiplicities $3, 2, 1$ in the characteristic polynomial of A. If the dimensions of the respective eigenspaces of a, b, c are $2, 1, 1$, what are the possible values of the degree of the minimal polynomial of A? Justify.

(b) Let B be a matrix over a field F. State a necessary and sufficient condition on the minimal polynomial of B for it to be triangulable but not diagonalizable over F. Is it true that if a matrix is triangular and diagonalizable then it must be diagonal? Justify.

(c) Find the Jordan form of the real matrix $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 7 & 8 & 0 \\ 0 & 0 & -4 & -5 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

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4. [20pts] Mark each of the following statements as TRUE or FALSE, giving full justification for your choice.

(a) If u and v are vectors in a finite-dimensional vector space such that $\{u\}^0 \subseteq \{v\}^0$ then u and v are linearly dependent.

(b) Let T be the linear operator on $\mathbb{C}^3$ whose representation in the standard ordered basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$ is the matrix $A = \begin{bmatrix} 1 & i & 0 \\ -1 & 2 & -i \\ 0 & 1 & 1 \end{bmatrix}$. Then the T-annihilator of the vector e_1 has degree 2.

(c) If f is a polynomial over $\mathbb{R}$ and E is a projection of a real vector space V, then $f(I - E) = aI + bE$ for some scalars a and b.

(d) If a linear operator T on an inner product space has an adjoint, then the adjoint need not be unique.