King Fahd University of Petroleum & Minerals Name
Department of Mathematics & Statistics
MATH 550/ Exam II/ 122 ID #

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1. **[20pts]** (a) Let T be the linear operator on $\mathbb{R}^2$ given by $T(x, y) = (2x + y, 2y)$ and let W_1 be the subspace of $\mathbb{R}^2$ spanned by the vector $e_1 = (1, 0)$. Prove that W_1 is T-invariant and that there is no T-invariant subspace W_2 of $\mathbb{R}^2$ such that $\mathbb{R}^2 = W_1 \oplus W_2$

(b) Let T be a linear operator on a finite-dimensional vector space V and let $\alpha \in V$. Prove that if W is a T-invariant subspace of V that contains α then $Z(\alpha, T) \subseteq W$. Is it true that $Z(\alpha, T)$ is equal to the intersection of all T-invariant subspaces of V that contain α? Justify

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2. **[24pts]** (a) Let A be a matrix over $\mathbb{R}$ such that $A^3 = A$. Is A diagonalizable? Justify.

(b) Let B be a 6×6 matrix with characteristic polynomial $f = (x^2 + 9)(x - 1)^4$ and minimal polynomial $p = (x^2 + 9)(x - 1)^2$.

(i) Find all possible rational forms of B over $\mathbb{R}$

(ii) Find all possible Jordan forms (up to order of Jordan blocks) of B over $\mathbb{C}$

(c) Let $C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$. Find a matrix P such that $P^{-1}CP$ is in Jordan form.

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3. **[20pts]** (a) Let E be a projection and T be a linear operator on a vector space V.

(i) Show that if $E + T = I$ then T is a projection and $\ker E = \operatorname{Im} T$

(ii) Show that if $\dim V \geq 3$, then V cannot contain a cyclic vector for E

(b) Let L be a linear operator on $\mathbb{R}^3$ whose matrix representation in the standard basis is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Prove that L has no cyclic vector.