King Fahd University of Petroleum & Minerals ID #
Department of Mathematics & Statistics
MATH 550/ Exam I/ 122 Name

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1. **[20pts]** Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $U : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformations defined by $T(x, y) = (x - y, x, 2x + y)$ and $U(x, y, z) = (z - 2y, x - 2y + z)$.

(a) Prove that UT is an isomorphism of $\mathbb{R}^2$ onto itself. Is TU one-one? Is it onto? Justify your answer.

(b) Let B be the standard basis of $\mathbb{R}^2$ and $B' = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Find $[T]_B^{B'}$.

(c) Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis for a vector space V and let $g : V \longrightarrow W$ be a linear transformation. Prove that $\{g(\alpha_1), \ldots, g(\alpha_n)\}$ is a linearly independent subset of W if and only if g is one-one.

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2. **[12pts]** Let V be a finite-dimensional vector space over a field F.

(a) Prove that if S is a nonempty subset of V, then $S^0 = (\text{span}(S))^0$.

(b) Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $v = (1, 0, -1, 2)$ and $w = (2, 3, 1, 1)$. Find all linear functionals $f : (x_1, x_2, x_3, x_4) \longmapsto c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ that are in W^0.

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3. **[20pts]** Let V, W be finite-dimensional vector spaces over a field F.

(a) Prove that if $T \in L(V, W)$ then $\ker(T^t) = (\operatorname{Im} T)^0$. Deduce that T is onto if and only if T^t is one-one.

(b) Prove that if T^t is onto then T is one-one.

(c) Prove that if T is an isomorphism then T^t is also an isomorphism.

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4. **[20pts]** (a) Is the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$ (over $\mathbb{R}$) diagonalizable? Justify your answer.

(b) Find all values of a for which the real matrix $\begin{bmatrix} 4 & 0 & 1 \\ 2 & a & 2 \\ 1 & 0 & 4 \end{bmatrix}$ is diagonalizable.

(c) Show that if an $n \times n$ matrix C over a field F is diagonalizable then its transpose C^t is also diagonalizable.