# King Fahd Univ. of Petroleum and Minerals
## Faculty of Sciences
## Department of Mathematics and Statistics

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MATH. 531-122

**Name:**

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**Prob. 1**

Let $E$ be a non-empty set and $\mathcal{C} \subset \mathcal{P}(E)$ a non-empty collection. Prove that there exists a smallest $\sigma$-algebra on $E$ containing $\mathcal{C}$.

**Prob. 2**

Let $f : E \to F$ be an application and $\mathcal{C} \subset \mathcal{P}(F)$. Prove that $\sigma_E(f^{-1}(\mathcal{C})) = f^{-1}(\sigma_F(\mathcal{C}))$.

**Prob. 3**

Let $(E, \Sigma, \mu)$ be a measure space, $A \in \Sigma$ and $B \subset E$ such that $A \Delta B \in \Sigma$ and $\mu(A \Delta B) = 0$. prove that $B \in \Sigma$ and $\mu(A \cap C) = \mu(B \cap C)$ for all $C \in \Sigma$.

**Prob. 4** Let $\mu^*$ be an outer measure on $E$ and $A \subset E$. Prove that the following are equivalent

(i) $A$ is $\mu^*$-measurable

(ii) For every $\varepsilon > 0$ there exist $A_1$ and $A_2$, $\mu^*$-measurable, $A_1 \subset A \subset A_2$ such that $\mu^*(A_2 \backslash A_1) < \varepsilon$

(iii) For every $P \subset A$ and $Q \subset A^c$ we have $\mu^*(P \cup Q) = \mu^*(P) + \mu^*(Q)$.

**Prob. 5**

Let $(E, \Sigma, \mu)$ be a finite measure space, $\mu^*$ the outer measure generated by $\mu$ and $A \subset E$. Prove that $A$ is $\mu^*$-measurable if and only if $\mu^*(E) = \mu^*(A) + \mu^*(A^c)$.

**Prob. 6**

Let $A \in \mathcal{L}$ be such that $0 < m(A) < \infty$. We define the application $f : \mathbf{R}^+ \to \mathbf{R}^+$ by $f(x) := m(A \cap (-x, x))$. Prove that

(i) $f$ is continuous and $f(\mathbf{R}^+)$ is equal to $[0, m(A)]$ or $[0, m(A))$

(ii) For all $c > 0$ there exist $\{A_n\}_{n \geq 1} \subset \mathcal{L}$, $A_n \subset A$, $n = 1, 2...$ such that $\sum_{n \geq 1} m(A_n) = cm(A)$.

**Prob. 7**

Let Let $\mu^* : \mathcal{P}(E) \to \bar{\mathbf{R}}^+$ be an outer measure. (a) Prove that $|\mu^*(A) - \mu^*(B)| \leq \mu^*(A\Delta B)$ for all $A, B \subset E$, such that $\mu^*(A) < \infty$ or $\mu^*(B) < \infty$.

(b) Let $A, B \subset E$ be such that $B$ is $\mu^*$-measurable, $B \subset A$ and $\mu^*(A) = \mu^*(B) < \infty$. Prove that $A$ is $\mu^*$-measurable.