MATH 470.1 (Term 122)

Homework Exercises 4 (Sects. 6.1-6.16) Due date: May 15, 2013

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1. Let S be the set of points (x, y) in $\mathbb{R}^2$ with $-2 \leq x < 3/2$, $0 \leq y \leq 4$ Determine all interior points of S, all boundary points of S, the closure of S, and whether S is open, closed, or neither open or closed. Is S connected? Is S bounded? Is S a domain?

2. Consider the following mixed boundary value problem for the Laplace equation in $\mathbb{R}^2$

$$\Delta u(x,y) = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{1}$$

with boundary conditions

$$u_y(x,0) = 0,\ u(x,b) = g(x),\ 0 \leq x \leq a, \quad u(0,y) = f(y),\ u(a,y) = 0,\ 0 \leq y \leq b,$$

where f and g are given suitably smooth functions.

a.) Assume that p is a solution of problem (1) for $g = 0$, and that q satisfies the problem (1) for $f = 0$. Verify that the sum $v = p + q$ is a solution of the full problem (1).

b.) Apply the method of separation of variables to obtain a solution of (1) for the specific boundary data.

$$f(y) = 5\cos\left(\frac{11}{2}\frac{\pi}{b}y\right) \text{ and } g = 0. \tag{2}$$

Check explicitly that your solution satisfies all conditions of problem (1).

3. Prove the representation theorem for functions in $\mathbb{R}^2$ by following the procedure that was detailed in the lecture for functions in $\mathbb{R}^3$:

a.) Derive Green's second identity

$$\iint_\Omega (u\Delta v - v\Delta u)dA = \oint_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds, \tag{3}$$

which holds for functions $u, v \in \mathcal{C}^2(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a piecewise smooth, closed boundary $\partial\Omega$.

b.) We know that the function

$$v(y) = \ln\frac{1}{|y - x|} \tag{4}$$

is harmonic in Ω with the exception of point x. Apply the identity (3) to a domain formed by removing a ball around x from Ω, with the function (4) inserted for v (note that all integrations are meant with respect to y). Then apply manipulations, limits, and simplifications analogous to the $\mathbb{R}^3$−case, until you arrive at the representation theorem for $\mathbb{R}^2$:

$$u(x) = \frac{1}{2\pi}\oint_{\partial\Omega}\left\{\frac{\partial u}{\partial n}(y)\ln\frac{1}{|y-x|} - u(y)\frac{\partial}{\partial n}\ln\frac{1}{|y-x|}\right\}ds - \frac{1}{2\pi}\iint_\Omega \Delta u(y)\ln\frac{1}{|y-x|}dA. \tag{5}$$

4. We know that the Laplace equation for the upper half-plane with boundary conditions on the x-axis

$$u_{xx} + u_{yy} = 0,\ u(x,0) = f(x),\ x \in \mathbb{R},\ y > 0, \tag{6}$$

has the solution

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(\xi)}{y^2 + (\xi - x)^2} d\xi. \tag{7}$$

a.) Consider the equation in the upper left quadrant with homogeneous boundary conditions on the positive y-axis:

$$v_{xx} + v_{yy} = 0,\ v(x,0) = g(x),\ v(0,y) = 0,\ x < 0,\ y > 0. \tag{8}$$

Use the result (6) to find an integral expression for the dependence of the solution of (8) on the function g, given on the negative x-axis.

b.) Now write a solution for the problem on the lower left quadrant:

$$w_{xx} + w_{yy} = 0,\ w(x,0) = h(x),\ w(0,y) = 0,\ x < 0,\ y < 0. \tag{9}$$

Q.1 :

$S = \{(x,y) \in \mathbb{R}^2,\ -2 \le x < 3/2,\ 0 \le y \le 4\}$

Interior of $S = \{(x,y) \in \mathbb{R}^2,\ -2 < x < 3/2,\ 0 < y < 4\}$

Boundary of $S = \{(-2,y), (\frac{3}{2},y),\ 0 \le y \le 4$
$(x,0), (x,4),\ -2 \le x \le \frac{3}{2}\}$

Closure of $S = \{(x,y) \in \mathbb{R}^2 /\ -2 \le x \le 3/2,$
$0 \le y \le 4\}$

S is not open, because points of the segment $x=-2$, $0 \le y \le 4$ are not interior points

S is not closed, because it does not contain the segment $x = 3/2$, $0 \le y \le 4$ of its boundary.

S is connected

S is bounded, $S \subset [-2, \frac{3}{2}] \times [0,4]$

S is not a domain, because not open.

Q.2

$$\begin{cases} \nabla^2 u(x,y) = 0, & 0 < x < a,\ 0 < y < b \\ u_y(x,0) = 0, & u(x,b) = g(x) \\ u(0,y) = f(y), & u(a,y) = 0 \end{cases}$$

a)

$$\begin{cases} \nabla^2 p(x,y) = 0 \\ p_y(x,0) = 0, & p(x,b) = 0 \\ p(0,y) = f(y), & p(a,y) = 0 \end{cases}$$

$$\begin{cases} \nabla^2 q(x,y) = 0 \\ q_y(x,0) = 0, & q(x,b) = g(x) \\ q(0,y) = 0, & q(a,y) = 0 \end{cases}$$

Of course

$\nabla^2(p+q) = \nabla^2 p + \nabla^2 q = 0$

$(p+q)_y(x,0) = p_y(x,0) + q_y(x,0) = 0$

$(p+q)(x,b) = p(x,b) + q(x,b) = g(x)$

$(p+q)(0,y) = p(0,y) + q(0,y) = f(x)$

$(p+q)(a,y) = p(a,y) + q(a,y) = 0$

Thus, $p+q$ is a solution to ①

b.) $p = XY$

$X''Y + XY'' = 0$

$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$\begin{cases} X'' + \lambda X = 0 \\ X(a) = 0 \end{cases}$ $\quad \begin{cases} Y'' - \lambda Y = 0 \\ Y'(0) = 0, \ Y(b) = 0 \end{cases}$

• $\lambda = 0$, $\quad Y = c_1 y + c_2$

$Y' = c_1 = 0 \ , Y(b) = c_2 = 0$

$\Rightarrow Y = 0$

• $\lambda = \alpha^2$, $\quad Y'' - \alpha^2 Y = 0$

$Y = c_1 \cosh \alpha y + c_2 \sinh \alpha y$

$Y' = c_1\alpha \sinh \alpha y + c_2 \alpha \cosh \alpha y$

$\left.\begin{array}{l} Y'(0) = 0 \Rightarrow c_2 = 0 \\ Y(b) = 0 \Rightarrow c_1 = 0 \end{array}\right\} \Rightarrow Y = 0$

• $\lambda = -\alpha^2$, $\quad Y'' + \alpha^2 Y = 0$

$Y = c_1 \cos \alpha y + c_2 \sin \alpha y$

$Y'(y) = -c_1 \alpha \sin \alpha y + c_2 \alpha \cos \alpha y$

$Y'(0) = 0 \Rightarrow c_2 = 0$

$Y(b) = 0 \Rightarrow \cos \alpha b = 0$

$\alpha_n b = \frac{\pi}{2} + n\pi \ , n = 0, 1, 2 \ldots$

$\alpha_n = \left(\frac{\pi}{2} + n\pi\right)\frac{1}{b}$

$Y_n = c \cos\left(\frac{\pi}{2} + n\pi\right)\frac{y}{b}$

$X'' - \alpha_n^2 X = 0$

$X = c_1 e^{\alpha_n x} + c_2 e^{-\alpha_n x}$

$X(a) = 0 \Rightarrow c_1 e^{\alpha_n a} + c_2 e^{-\alpha_n a} = 0$

$c_1 = -c_2 e^{-2\alpha_n a}$

$\Rightarrow X(x) = -c_2 e^{-2\alpha_n a} e^{\alpha_n x} + c_2 e^{-\alpha_n x}$

$= -c_2 e^{-\alpha_n a}\left(e^{\alpha_n (x-a)} - e^{-\alpha_n (x-a)}\right)$

$= c_n \sinh \alpha_n (x-a)$

$\Rightarrow p(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\pi}{2} + n\pi\right)\frac{y}{b} \sinh\left(\frac{\pi}{2} + n\pi\right)\frac{(x-a)}{b}$

$p(0,y) = f(y) \Rightarrow$

$f(y) = -\sum_{n=0}^{\infty} A_n \sinh\left(\frac{\pi}{2} + n\pi\right)\frac{a}{b} \cos\left(\frac{\pi}{2} + n\pi\right)\frac{y}{b}$

$\Rightarrow -A_n \sinh\left(\frac{\pi}{2} + n\pi\right)\frac{a}{b} = \frac{2}{b}\int_0^b f(\xi) \cos\left(\frac{\pi}{2} + n\pi\right)\frac{\xi}{b}\, d\xi$

$A_n = -\frac{2}{b \sinh\left(\frac{\pi}{2} + n\pi\right)\frac{a}{b}} \int_0^b f(\xi) \cos\left(\frac{\pi}{2} + n\pi\right)\frac{\xi}{b}\, d\xi$

When $f(y) = 5\cos\left(\frac{11}{2}\frac{\pi}{b} y\right)$

$\int_0^b \cos\left(\frac{11}{2}\frac{\pi}{b}\xi\right) \cos\left(\frac{\pi}{2} + n\pi\right)\frac{\xi}{b}\, d\xi =$

$\frac{1}{2}\int_0^b \left[\cos \frac{\pi}{b}(n+6)\xi + \cos \frac{\pi}{b}(n-5)\xi\right] d\xi$

$= \frac{1}{2}\left[\frac{\sin \frac{\pi}{b}(n+6)\xi}{\frac{\pi}{b}(n+6)} + \frac{\sin \frac{\pi}{b}(n-5)\xi}{\frac{\pi}{b}(n-5)}\right]_0^b$

$= \begin{cases} 0 & \text{if } n \neq 5 \\ \frac{b}{2} & \text{if } n = 5 \end{cases}$

Thus, $A_n = \begin{cases} 0 & , n \neq 5 \\ -\dfrac{5}{\sinh \frac{a}{b}\pi \frac{11}{2}} & , n = 5 \end{cases}$

$\Rightarrow P(x,y) = A_5 \cos \frac{11}{2}\pi \frac{y}{b} \cdot \sinh \frac{11}{2}\frac{\pi}{b}(x-a)$

$$\boxed{P(x,y) = -\frac{5}{\sinh \frac{11}{2}\frac{a}{b}\pi} \cos \frac{11}{2}\frac{\pi}{b} y \, \sinh \frac{11}{2}\frac{\pi}{b}(x-a)}$$

Q.3

a.) The Green theorem is

$$\oint_{\partial\Omega} u \nabla v \cdot \vec{n}\, ds = \iint_{\Omega} \nabla \cdot (u \nabla v)\, dA,$$

where $\vec{n}$ is outer normal vector.

By definition $\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \nabla^2 v$

and

$$\oint_{\partial\Omega} u \nabla v \cdot \vec{n} = \oint_{\partial\Omega} u \frac{\partial v}{\partial n}\, ds$$

$$\Rightarrow \oint_{\partial\Omega} u \frac{\partial v}{\partial n}\, ds = \iint_{\Omega} \nabla u \cdot \nabla v\, dA + \iint_{\Omega} u \nabla^2 v\, dA \quad (1)$$

We also have

$$\oint_{\partial\Omega} v \frac{\partial u}{\partial n}\, ds = \iint_{\Omega} \nabla v \cdot \nabla u\, dA + \iint_{\Omega} v \nabla^2 u\, dA \quad (2)$$

Substracting (1) from (2), we find

$$\iint_{\Omega} (u \nabla^2 v - v \nabla^2 u)\, dA = \oint_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) ds \quad (3)$$

b.) $v(y) = \ln \frac{1}{|y-x|}$ is harmonic in $\Omega - \{x\}$.

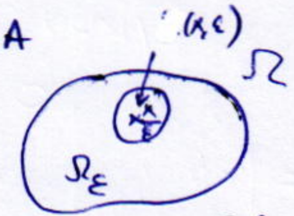
Let $D(x,\varepsilon)$ a disk centred at x of radius ε such that $\overline{D(x,\varepsilon)} \subset \Omega$.

We set $\Omega_\varepsilon = \Omega - \overline{D}(x,\varepsilon)$

Ω_ε is open and connected, so it is a domain.

Now, we apply formula (3) on Ω_ε and $v(y) = \ln \frac{1}{|y-x|}$

$$\Rightarrow \oint_{\partial\Omega_\varepsilon} \left(u \frac{\partial}{\partial n} \ln \frac{1}{|y-x|} - \ln \frac{1}{|y-x|} \frac{\partial u}{\partial n}\right) ds = -\iint_{\Omega_\varepsilon} \ln \frac{1}{|y-x|} \nabla^2 u\, dA$$



$\partial\Omega_\varepsilon = \partial\Omega \cup C(x,\varepsilon)$

Thus, we have

$$\oint_{\partial\Omega} \left[u(y) \frac{\partial}{\partial n} \ln \frac{1}{|y-x|} - \ln \frac{1}{|y-x|} \frac{\partial u(y)}{\partial n}\right] ds$$

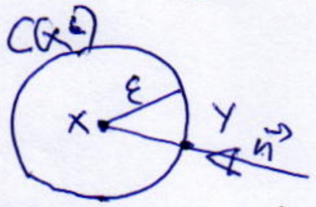
$$+ \oint_{C(x,\varepsilon)} \left[u(y) \frac{\partial}{\partial n} \ln \frac{1}{|y-x|} - \ln \frac{1}{|y-x|} \frac{\partial u(y)}{\partial n}\right] ds$$

$$= -\iint_{\Omega_\varepsilon} \ln \frac{1}{|y-x|} \nabla^2 u\, dA.$$

Now, let $\varepsilon \to 0$

The term $\oint_{\partial\Omega}$ doesn't depend on ε, we can pass to the limit $\varepsilon \to 0$ in this term.

The term $\oint_{C(x,\varepsilon)}$ depends on ε.



for $y \in C(x,\varepsilon)$, $|y-x| = \varepsilon$

$x=(x_1, x_2)$, $y=(y_1, y_2)$

$$\vec{n}(y) = -\frac{(y-x)}{|y-x|} = -\frac{(y_1-x_1)\vec{i} + (y_2-x_2)\vec{j}}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}}$$

$$\ln\frac{1}{|y-x|} = -\frac{1}{2}\ln\left[(y_1-x_1)^2+(y_2-x_2)^2\right]$$

$$\nabla \ln\frac{1}{|y-x|} = -\frac{(y_1-x_1)\vec{i} + (y_2-x_2)\vec{j}}{(y_1-x_1)^2+(y_2-x_2)^2}$$

$$\frac{\partial}{\partial n}\ln\frac{1}{|y-x|} = \nabla \ln\frac{1}{y-x} \cdot \vec{n}(y)$$

$$= \frac{(y_1-x_1)^2+(y_2-x_2)^2}{\left[(y_1-x_1)^2+(y_2-x_2)^2\right]^{3/2}}$$

$$= \frac{1}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}} = \frac{1}{|y-x|} = \frac{1}{\varepsilon}$$

$$\Rightarrow \oint_{C(x,\varepsilon)} \left[u(y)\frac{\partial}{\partial n}\ln\frac{1}{|y-x|} - \ln\frac{1}{|y-x|}\frac{\partial}{\partial n}u(y)\right] dy$$

$$= \int_{C(x,\varepsilon)} \left[\frac{1}{\varepsilon}u(y) - \ln\frac{1}{|y-x|}\frac{\partial}{\partial n}u(y)\right] dy$$

$$= \int_{C(x,\varepsilon)} \frac{1}{\varepsilon}u(x)\, dy + \int_{C(x,\varepsilon)} \frac{1}{\varepsilon}\left(u(y)-u(x)\right) dy$$

$$- \int_{C(x,\varepsilon)} \ln\frac{1}{|y-x|}\frac{\partial u(y)}{\partial n} dy$$

But, $$\int_{C(x,\varepsilon)} \frac{1}{\varepsilon}u(x)\, dy = \frac{1}{\varepsilon}u(x)(2\pi\varepsilon) = 2\pi u(x)$$

$$\left|\int_{C(x,\varepsilon)}\left[\frac{1}{\varepsilon}(u(y)-u(x))\, dy - \ln\frac{1}{|y-x|}\frac{\partial}{\partial n}u(y)\right]dy\right|$$

$$\leq \frac{1}{\varepsilon}(2\pi\varepsilon)\max_{y\in C(x,\varepsilon)}|u(y)-u(x)|$$

$$+ (\ln\varepsilon)(2\pi\varepsilon)\max_{y\in C(x,\varepsilon)}\left|\frac{\partial}{\partial n}u(y)\right|$$

When $\varepsilon \to 0$, we have $\lim_{\varepsilon\to 0}\max_{y\in C(x,\varepsilon)}|u(y)-u(x)| = 0$

and $\varepsilon\ln\varepsilon = 0$

$$\Rightarrow \lim_{\varepsilon\to 0}\int_{C(x,\varepsilon)}\left[\frac{1}{\varepsilon}(u(y)-u(x)) - \ln\frac{1}{|y-x|}\frac{\partial u(y)}{\partial n}\right]dy = 0$$

Thus, we find

$$\oint_{\partial\Omega}\left[u(y)\frac{\partial}{\partial n}\ln\frac{1}{|y-x|} - \ln\frac{1}{|y-x|}\frac{\partial}{\partial n}u(y)\right] dy + 2\pi u(x)$$

$$= -\iint_{\Omega}\ln\left|\frac{1}{y-x}\right|\nabla^2 u\, dA$$

$$\Rightarrow u(x) = \frac{1}{2\pi}\oint_{\partial\Omega}\left[\ln\frac{1}{|y-x|}\frac{\partial}{\partial n}u(y) - u(y)\frac{\partial}{\partial n}\ln\frac{1}{|y-x|}\right]dy$$

$$-\frac{1}{2\pi}\iint_{\Omega}\ln\frac{1}{|y-x|}\nabla^2 u\, dA$$

since,

$$\lim_{\varepsilon\to 0} \iint_{\Omega_\varepsilon} \ln \frac{1}{|Y-X|} \nabla^2 u \, dA = \iint_{\Omega} \ln \frac{1}{|Y-X|} \nabla^2 u \, dA$$

because this improper integral converge.

$\boxed{\text{Q. 4}}$

The solution of the Laplace's equation

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in \mathbb{R},\ y > 0 \\ u(x,0) = f(x) \end{cases}$$

is $u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(\xi)}{y^2 + (\xi - x)^2} d\xi$

a) Consider now the problem

$$\begin{cases} v_{xx} + v_{yy} = 0, & x < 0,\ y > 0 \\ v(x,0) = g(x) \\ v(0,y) = 0 \end{cases} \quad (1)$$

let $\tilde{g}(x) = \begin{cases} p(x), & x \geq 0 \\ g(x), & x < 0 \end{cases}$

and define

$$\begin{cases} v_{xx} + v_{yy} = 0, & x \in \mathbb{R},\ y > 0 \\ v(x,0) = \tilde{g}(x) \\ v(0,y) = 0. \end{cases}$$

This new problem has the solution

$$v(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y \tilde{g}(\xi)}{y^2 + (\xi - x)^2} d\xi$$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{0} \frac{y g(\xi)}{y^2 + (\xi - x)^2} d\xi + \int_{0}^{\infty} \frac{y p(\xi)}{y^2 + (\xi - x)^2} d\xi \right]$$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{0} \frac{y g(\xi)}{y^2 + (\xi - x)^2} d\xi + \int_{-\infty}^{0} \frac{y p(-\xi)}{y^2 + (\xi + x)^2} d\xi \right]$$

$$v(0,y) = \frac{1}{\pi} \int_{-\infty}^{0} \left[\frac{y g(\xi)}{y^2 + \xi^2} + \frac{y p(-\xi)}{y^2 + \xi^2} \right] d\xi$$

$$v(0,y) = 0 \Rightarrow \quad p(-\xi) = -g(\xi)$$

$$p(x) = -g(-x), \quad x \geq 0$$

Thus,

$$v(x,y) = \frac{1}{\pi} \left[\int_{-\infty}^{0} \frac{y g(\xi)}{y^2 + (\xi - x)^2} d\xi - \int_{-\infty}^{0} \frac{y g(\xi)}{y^2 + (\xi + x)^2} d\xi \right]$$

$$= \frac{1}{\pi} \int_{-\infty}^{0} \left(\frac{y}{y^2 + (\xi - x)^2} - \frac{y}{y^2 + (\xi + x)^2} \right) g(\xi) \, d\xi$$

This is also the solution for (1), $x < 0$, $y > 0$.

b) Consider now the problem

$$\begin{cases} w_{xx} + w_{yy} = 0, & x < 0,\ y < 0 \\ w(x,0) = h(x) \\ w(0,y) = 0 \end{cases} \quad (2)$$

Set $w(x,y) = v(x,-y)$, $y < 0$

$w_{xx}(x,y) = v_{xx}(x,-y)$

$w_y(x,y) = -v_y(x,-y)$

$w_{yy}(x,y) = v_{yy}(x,-y)$

Thus, $v(x,-y)$ satisfies the problem

$$\begin{cases} v_{xx}(x,-y) + v_{yy}(x,-y) = 0 \\ v(x,0) = h(x) \\ v(0,-y) = 0 \end{cases} \qquad (3)$$

Since $y<0$, we have that $-y>0$

The solution of (3) is

$$v(x,-y) = \frac{1}{\pi} \int_{-\infty}^{0} -y \left(\frac{1}{y^2+(\xi-x)^2} - \frac{1}{y^2+(\xi+x)^2} \right) h(\xi)\, d\xi$$

$$w(x,y) = v(x,-y)$$

$$\Rightarrow \quad w(x,y) = -\frac{y}{\pi} \int_{-\infty}^{0} \left(\frac{1}{y^2+(\xi-x)^2} - \frac{1}{y^2+(\xi+x)^2} \right) h(\xi)\, d\xi,$$

for $x<0$, $y<0$