MATH 470.1 (Term 122)

Homework Exercises 3 (Sects. 5.6-5.8) Due date: April 29, 2013

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1. Assume that u satisfies an inhomogeneous 1-D wave equation on an interval $[0, L]$

$$u_{tt} = c^2 u_{xx} + f(x,t), \text{ for } 0 < x < L, t > 0, \tag{1}$$

with given inhomogeneity f and constant phase velocity $c > 0$.

a.) Show that the following integral relationship holds for any interval $[a, b]$ with $0 < a, b < L$:

$$\frac{d}{dt}\int_a^b \frac{1}{2}(u_t^2 + c^2 u_x^2)dx = c^2 u_t u_x|_a^b + \int_a^b f u_t dx \tag{2}$$

(Hint: Multiply the wave equation by u_t, integrate over $[a, b]$, then use integration by parts on a suitable term. Note that $\partial_t(u_t^2) = 2u_t u_{tt}$).

b.) Show that the IBVP (1), with

$$u(0,t) = u(L,t) = 0, \; u(x,0) = g(x), \text{ for } 0 \le x \le L, \; t \ge 0,$$

with g a given function, can have only one solution. (Hint: Assume there are two solutions; take the difference and determine the conditions (PDe & boundary/initial values) that are satisfied by the function. Evaluate the result a.) above for the difference function).

2. Consider the following heat problem in dimensionless variables

$$u_t = u_{xx} + \frac{\pi^2}{4}u - b, \quad 0 < x < 1, \quad t > 0 \tag{3}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0 \tag{4}$$

$$u(x,0) = u_0, \quad 0 < x < 1. \tag{5}$$

a.) Derive the equilibrium solution $u_E(x)$, that is, when $u_t = 0$.

b.) Using $u_E(x)$, transform the given heat problem for $u(x,t)$ into the following problem for a function $v(x,t)$:

$$v_t = v_{xx} + \frac{\pi^2}{4}v, \quad 0 < x < 1, \quad t > 0 \tag{6}$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0 \tag{7}$$

$$v(x,0) = f(x), \quad 0 < x < 1, \tag{8}$$

where $f(x)$ will be determined by the transformation.

c.) For an appropriate value of α show that the transformation $w(x,t) = e^{\alpha t}v(x,t)$ further simplifies the problem to

$$w_t = w_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{9}$$

$$w(0,t) = 0, \quad w(1,t) = 0, \quad t > 0 \tag{10}$$

$$w(x,0) = f(x), \quad 0 < x < 1. \tag{11}$$

d.) Deduce the solution u of (3)-(5).

3. Suppose u is a bounded solution of the initial value problem

$$u_t = Ku_{xx} \text{ for } -\infty < x < \infty,\ t > 0,\ u(x,0) = f(x) \text{ for } -\infty < x < \infty \qquad (12)$$

for a given continuous function f, with $u(x,t) \to 0$ uniformly in t as $x \to \pm\infty$. K is a positive constant. Show that

$$|u(x,t)| \le \max_{x\in\mathbb{R}} |f(x)| \qquad (13)$$

for all x, t with $-\infty < x < \infty$ and $t > 0$. (Hint: apply the weak maximum principle on an interval $x \in [-a, a]$ for $t \ge 0$, and take the limit $a \to \infty$).

Q1:

$$u_{tt} = c^2 u_{xx} + f(x,t) \quad (1)$$

a.) We multiply (1) by u_t

$$u_{tt} u_t = c^2 u_{xx} u_t + f(x,t)\, u_t. \quad (2)$$

We integrate (2) on $x \in [a,b]$.

$$\int_a^b u_{tt} u_t \, dx = c^2 \int_a^b u_{xx} u_t \, dx + \int_a^b f(x,t) u_t \, dx$$

Now, $\int_a^b u_{tt} u_t \, dx = \frac{1}{2}\frac{d}{dt}\int_a^b u_t^2 \, dx$

We integrate by parts in $\int_a^b u_{xx} u_t \, dx$.

$$\int_a^b u_{xx} u_t \, dx = \left[u_x u_t\right]_a^b - \int_a^b u_x u_{xt} \, dx$$

$$= \left[u_x u_t\right]_a^b - \frac{1}{2}\frac{d}{dt}\int_a^b u_x^2 \, dx$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\int_a^b (u_t^2 + c^2 u_x^2) = c^2\left[u_x u_t\right]_a^b + \int_a^b f(x,t) u_t \, dx \quad (3)$$

b.) Assume there are two solutions u_1 and u_2.

Let $w = u_1 - u_2$

Thus, w satisfies the IVP

$$\begin{cases} w_{tt} = c^2 w_{xx} \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = 0, \quad 0 < x < L,\ t > 0 \end{cases}$$

Apply the estimate (3) from previous question, we find

$$\frac{d}{dt}\int_0^L \frac{1}{2}(w_t^2 + c^2 w_x^2)\, dx = \left[c^2 w_t w_x\right]_0^L$$

$w(x,0) = 0 \Rightarrow w_x(x,0) = 0, \ 0 \le x \le L$

$w(0,t) = 0 \Rightarrow w_t(0,t) = 0$

$w(L,t) = 0 \Rightarrow w_t(L,t) = 0$

$$\Rightarrow c^2\left[w_t(L,t) w_x(L,t) - w_t(0,t) w_x(0,t)\right] = 0$$

$$\frac{d}{dt}\int_0^L \frac{1}{2}(w_t^2 + w_x^2)\, dx = 0$$

$$\int_0^L (w_t^2 + w_x^2)\, dx = \int_0^L \left[w_t^2(x,0) + w_x^2(x,0)\right] dx$$

$$= 0$$

$\Rightarrow W_t = 0$ and $W_x = 0$

$\Rightarrow W(x,t) = $ Constant

$W(0,t) = W(L,t) = 0$

$\Rightarrow$ Const $= 0$

$\Rightarrow W(x,t) = 0$

Thus, $u_1 = u_2$.

Q2: a) We solve

$$u_{xx} + \frac{\pi^2}{4} u = b$$

$$\Rightarrow u(x) = c_1 \cos\frac{\pi}{2}x + c_2 \sin\frac{\pi}{2}x + \frac{4b}{\pi^2}$$

$u(0) = u(1) = 0$

$$\Rightarrow c_1 = -\frac{4b}{\pi^2}$$

$$\frac{-4b}{\pi^2} \underbrace{\cos\frac{\pi}{2}}_{=0} + c_2 \underbrace{\sin\frac{\pi}{2}}_{1} = -\frac{4b}{\pi^2}$$

$$\Rightarrow c_2 = -\frac{4b}{\pi^2}$$

$$\Rightarrow u_E(x) = \frac{-4b}{\pi^2}\left(\cos\frac{\pi}{2}x + \sin\frac{\pi}{2}x - 1\right)$$

b) let $v = u - u_E$

$$\Rightarrow \begin{cases} v_t = v_{xx} + \frac{\pi^2}{4} v \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = \underbrace{u_0 + \frac{4b}{\pi^2}\left(\cos\frac{\pi}{2}x + \sin\frac{\pi}{2}x - 1\right)}_{f(x)} \end{cases}$$

c) let $w = e^{\alpha t} v(x,t)$

$$W_t = (\alpha v + v_t) e^{\alpha t} \quad (1)$$

$$W_x = e^{\alpha t} v_x$$

$$W_{xx} = e^{\alpha t} v_{xx}$$

From (1) $\Rightarrow v_t = e^{-\alpha t} W_t - \alpha v$

$$v_t = v_{xx} + \frac{\pi^2}{4} v \Rightarrow e^{-\alpha t} W_t - \alpha v = e^{-\alpha t} W_{xx} + \frac{\pi^2}{4} v$$

$$\Rightarrow W_t = W_{xx} + e^{\alpha t} \underbrace{\left(\alpha + \frac{\pi^2}{4}\right)}_{=0} v$$

let $\alpha = -\pi^2/4$

$$\Rightarrow \begin{cases} W_t = W_{xx} \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = v(x,0) = f(x) \end{cases}$$

d) $w(x,t) = X(x)T(t)$

$$XT' = X''T \Leftrightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(1) = 0 \end{cases} \qquad T' + \lambda T = 0$$

$\lambda = 0$ and $\lambda < 0 \Rightarrow X = 0$

$\lambda > 0$, $\lambda = \alpha^2 \Rightarrow X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$

$X(0) = 0 \Rightarrow c_1 = 0$

$X(1) = 0 \Rightarrow \sin\alpha = 0$, $\alpha = n\pi$

$\Rightarrow X_n(x) = c \sin n\pi x$

$T' + (n\pi)^2 T = 0$

$T_n(t) = c\, e^{-n^2\pi^2 t}$

$$\Rightarrow W(x,t) = \sum_{n=1}^{\infty} B_n \sin n\pi x \, e^{-n^2\pi^2 t}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x$$

$$\Rightarrow B_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx$$

$$= 2\int_0^1 \left(u_0 + \frac{4b}{\pi^2}\cos\frac{\pi}{2}x + \frac{4b}{\pi^2}\sin\frac{\pi}{2}x - \frac{4b}{\pi^2}\right)\sin n\pi x\,dx$$

$$= 2\left(u_0 - \frac{4b}{\pi^2}\right)\left[\frac{-\cos n\pi x}{n\pi}\right]_0^1 + \frac{8b}{\pi^2}\int_0^1 \cos\frac{\pi}{2}x \sin n\pi x\,dx$$

$$+ \frac{8b}{\pi^2}\int_0^1 \sin\frac{\pi}{2}x \sin n\pi x\,dx$$

$$= \frac{2}{n\pi}\left(u_0 - \frac{4b}{\pi^2}\right)\left(1-(-1)^n\right)$$

$$+ \frac{4b}{\pi^2}\int_0^1 \left[\sin\left(\frac{\pi}{2}+n\pi\right)x - \sin\left(\frac{\pi}{2}-n\pi\right)x\right]dx$$

$$+ \frac{4b}{\pi^2}\int_0^1 \left[-\cos\left(\frac{\pi}{2}+n\pi\right)x + \cos\left(\frac{\pi}{2}-n\pi\right)x\right]dx$$

$$= \frac{2}{n\pi}\left(u_0 - \frac{4b}{\pi^2}\right)\left(1-(-1)^n\right)$$

$$+ \frac{4b}{\pi^2}\left[\frac{-\cos\left(\frac{\pi}{2}+n\pi\right)x}{\frac{\pi}{2}+n\pi} + \frac{\cos\left(\frac{\pi}{2}-n\pi\right)x}{\frac{\pi}{2}-n\pi}\right]_0^1$$

$$+ \frac{4b}{\pi^2}\left[\frac{-\sin\left(\frac{\pi}{2}+n\pi\right)x}{\frac{\pi}{2}+n\pi} + \frac{\sin\left(\frac{\pi}{2}-n\pi\right)x}{\frac{\pi}{2}-n\pi}\right]_0^1$$

$$= \frac{2}{n\pi}\left(u_0 - \frac{4b}{\pi^2}\right)\left(1-(-1)^n\right)$$

$$+ \frac{4b}{\pi^2}\left(\frac{-(-1)^n}{\frac{\pi}{2}+n\pi} + \frac{(-1)^n}{\frac{\pi}{2}-n\pi}\right)$$

$$= + \frac{4b}{\pi^2}\left(\frac{1}{\frac{\pi}{2}+n\pi} - \frac{1}{\frac{\pi}{2}-n\pi}\right)$$

$$= \frac{2}{n\pi}\left(u_0 - \frac{4b}{\pi^2}\right)\left(1-(-1)^n\right)$$

$$+ \frac{32b}{\pi^3}\frac{1}{(1-4n^2)}\left(-1+(-1)^n\right)$$

So, $B_n = \begin{cases} 0, & \text{if } n = 2p \\ \frac{4}{n\pi}\left(u_0 - \frac{4b}{\pi^2}\right) - \frac{64b}{\pi^3(1-4n^2)}, & \text{if } n = 2p+1 \end{cases}$

$$\Rightarrow W(x,t) = \sum_{p=0}^{\infty} \frac{4}{\pi}\left(\frac{u_0 - \frac{4b}{\pi^2}}{2p+1} - \frac{8b}{\pi^2(1-4(2p+1)^2)}\right) \times \sin(2p+1)\pi x\, e^{-(2p+1)^2\pi^2 t}$$

and

$$u(x,t) = e^{\frac{\pi^2}{4}t} \sum_{p=0}^{\infty} \frac{4}{\pi}\left(\frac{u_0 - \frac{4b}{\pi^2}}{2p+1} - \frac{8b}{\pi^2(1-4(2p+1)^2)}\right) \times \sin(2p+1)\pi x\, e^{-(2p+1)^2\pi^2 t}$$

$$+ \frac{4b}{\pi^2}\left(1 - \cos\frac{\pi}{2}x - \sin\frac{\pi}{2}x\right)$$

Q.3 Let $\Omega_a = [-a,a]$

$$\begin{cases} u_t = K u_{xx} & -a < x < a,\ t > 0 \\ u(x,0) = f(x) \end{cases}$$

Let $T > 0$. $u(x,t)$ is continuous on $[-a,a] \times [0,T]$.

From the weak maximum principle, $u(x,t)$ assumes its maximum value at $t=0$ or $x=-a$ or $x=-a$

Thus, $|u(x,t)| \leq \max\{|f(x)|, u(a,t), u(-a,t)\}$

for $(x,t) \in [-a,a] \times [0,T]$.

Since $f(x)$ is continuous, we have that,

$$|f(x)| \leq \max_{x \in R} |f(x)|$$

for every $x \in [-a,a]$.

Now, $u(x,t) \to 0$ uniformly in time, when $x \to \pm\infty$.

Thus, there exists $M > 0$ such that

$$|u(x,t)| \leq \max_{x \in \mathbb{R}} |f(x)|,$$

for $|x| \geq M$

$|a| \geq M$, then

$$|u(x,t)| \leq \max_{x \in \mathbb{R}} |f(x)|,$$

for $|x| \geq a$

This estimate is uniform with respect to a.

Now let $|a| \to \infty$

$$\Rightarrow |u(x,t)| \leq \max_{x \in R} |f(x)|$$

for every $x \in (-\infty,\infty)$, $t > T$.

T is arbitrary and the estimate is also uniform with respect to T.

$$\Rightarrow |u(x,t)| \leq \max_{x \in R} |f(x)|,$$

for every $-\infty < x < \infty$, $t > 0$