MATH 470.1 (Term 122)

Homework Exercises 2 (Sects. 2.6-2.7, 4.1-4.8) **Due date: March 18, 2013**

---

1. Consider the Cauchy problem for the PDE

$$u_{xx} - 6u_{yy} - 2u_x = 0 \tag{1}$$

with Cauchy data on the line Γ given by $y = 5x$:

$$u(x,5x) = \sin x, \quad (n\cdot\nabla)u|_(x,5x) = \frac{1}{\sqrt{26}}(5u_x(x,5x) - u_y(x,5x)) = \frac{1}{\sqrt{26}}x^2.$$

Here $n = \frac{1}{\sqrt{26}}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ is a normal vector perpendicular to Γ.

Use Cauchy data and the PDE to compute and write the Taylor expansion of the solution around the origin up to second order.

2. Consider a 1-D wave equation on the entire spatial axis

$$u_{tt} = 9u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2}$$

with initial data

$$u_t(x,0) = 0, \quad u(x,0) = \begin{cases} \sin^2(\pi x) & \text{for } -2 < x < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Write the solution of the Cauchy problem as a sum of forward and backward waves. Based on the splitting, graph (qualitatively) the solution a times $t = 0$, $t = 1/6$, $t = 1/3$ and $t = 2/3$.

3. Given the 1-D wave equation on the positive real axis

$$u_{xx} - \frac{1}{4}u_{tt} = 0, \quad 0 < x < \infty, \quad t > 0, \tag{4}$$

the initial data $u(x,0) = x\sin(\pi x)$, $u_t(x,0) = x(1-x)$, for $x > 0$, and the homogeneous boundary condition $u(0,t) = 0$, evaluate the (piecewise defined) solution of the initial-boundary value problem. Verify explicitly that your solution satisfies the initial and boundary conditions, and that the function is continuous.

4. Evaluate the solution of the Cauchy problem for the inhomogeneous 1-D wave equation on the entire spatial axis

$$u_{tt} = 9u_{xx} + 3xt, \quad -\infty < x < \infty, \quad t > 0, \tag{5}$$

with initial data $u(x,0) = 3\sin x$ and $u_t(x,0) = x$. Check explicitly that your function satisfies the equation (5) and the initial conditions.

5. Apply the method of separation of variables to obtain a formal Fourier series solution of the telegraph equation

$$u_{tt} + Au_t + Bu = c^2 u_{xx}, \quad \text{for } 0 < x < L, \quad t > 0, \tag{6}$$

with boundary conditions

$$u(0,t) = u(L,t) = 0 \text{ for } t > 0, \quad u(x,0) = \varphi(x),\ u_t(x,0) = 0 \text{ for } 0 < x < L, \tag{7}$$

where A, B and c are positive constants with $A^2L^2 < 4(BL^2 + c^2\pi^2)$.

Ex1: The characteristic equations are $\frac{dy}{dx} = \pm\sqrt{6}$

$\iff y = \pm\sqrt{6}x$ are the characteristic curves.

So, $\Gamma: y=5x$ is not a characteristic curve.

Thus the Cauchy problem has a unique solution that can be expand as a Taylor series about the origin $(0,0)$, that is,

$$u(x,y) = u(0,0) + x u_x(0,0) + y u_y(0,0) + \frac{1}{2}\left(x^2 u_{xx}(0,0) + 2xy u_{xy}(0,0) + y^2 u_{yy}(0,0)\right) + \cdots$$

Now, we can compute $u(0,0)$, but we don't know how to compute $u_x(0,0)$, $u_y(0,0)$, etc....

To overcome this difficulty, we introduce new variables.

Let $\xi = y-5x$ and $\eta = x+y$

$$J = \begin{vmatrix} -5 & 1 \\ 1 & 1 \end{vmatrix} = -6 \neq 0$$

So, on Γ, $\xi = 0$

Let $u(x,y) = \varphi(\xi,\eta)$

$$u_x = \varphi_\xi \xi_x + \varphi_\eta \eta_x = -5\varphi_\xi + \varphi_\eta$$

$$u_{xx} = (-5\varphi_\xi + \varphi_\eta)_\xi \xi_x + (-5\varphi_\xi + \varphi_\eta)_\eta \eta_x$$
$$= 25\varphi_{\xi\xi} - 10\varphi_{\xi\eta} + \varphi_{\eta\eta}$$

$$u_y = \varphi_\xi \xi_y + \varphi_\eta \eta_y = \varphi_\xi + \varphi_\eta$$

$$u_{yy} = (\varphi_\xi + \varphi_\eta)_\xi \xi_y + (\varphi_\xi + \varphi_\eta)_\eta \eta_y$$
$$= \varphi_{\xi\xi} + 2\varphi_{\xi\eta} + \varphi_{\eta\eta}$$

The PDE becomes

$$(25\varphi_{\xi\xi} - 10\varphi_{\xi\eta} + \varphi_{\eta\eta}) - 6(\varphi_{\xi\xi} + 2\varphi_{\xi\eta} + \varphi_{\eta\eta}) - 2(-5\varphi_\xi + \varphi_\eta) = 0,$$

$$19\varphi_{\xi\xi} - 22\varphi_{\xi\eta} - 5\varphi_{\eta\eta} + 10\varphi_\xi - 2\varphi_\eta = 0$$

We have $x = \frac{\eta - \xi}{6}$, $y = \frac{\xi + 5\eta}{6}$

We parametrize Γ using arc length

$x(s) = \frac{s}{\sqrt{26}}$ and $y(s) = \frac{5s}{\sqrt{26}}$

$x'(s)^2 + y'(s)^2 = 1$.

$$\Rightarrow \varphi(x(s), y(s)) = \sin\left(\frac{s}{\sqrt{26}}\right) = f(s)$$

$$\varphi_n(x(s), y(s)) = \frac{1}{\sqrt{26}} \frac{s^2}{26} = g(s)$$

$$\varphi_\xi = \frac{1}{J}\left[(x'f' - y'g)\eta_y - (x'g + y'f')\eta_x\right]$$

$$\varphi_\eta = \frac{1}{J}\left[(x'g + y'f')\xi_x - (x'f' - y'g)\xi_y\right]$$

$$\Rightarrow \varphi_\xi = \frac{s^2}{26^2} + \frac{1}{39}\cos\left(\frac{s}{\sqrt{26}}\right)$$

$$\varphi_\eta = \frac{1}{6}\cos\left(\frac{s}{\sqrt{26}}\right)$$

On Γ: $\xi = 0$, $\frac{\eta}{6} = \frac{s}{\sqrt{26}}$, $s = \frac{\sqrt{26}}{6}\eta$

$$\Rightarrow \varphi_\xi(0,\eta) = \frac{\eta^2}{26.36} + \frac{1}{39}\cos\left(\frac{\eta}{6}\right)$$

$$\varphi_\eta(0,\eta) = \frac{1}{6}\cos\left(\frac{\eta}{6}\right)$$

$$\varphi_{\xi\eta}(0,\eta) = \frac{\eta}{13.36} - \frac{1}{6.39}\sin\left(\frac{\eta}{6}\right)$$

$$\varphi_{\eta\eta}(0,\eta) = -\frac{1}{36}\sin\left(\frac{\eta}{6}\right)$$

$$\varphi_{\xi\xi}(0,\eta) = \frac{1}{19}\left[22\varphi_{\xi\eta} + 5\varphi_{\eta\eta} - 10\varphi_\xi + 2\varphi_\eta\right]$$

$$\varphi_{\xi\xi}(0,\eta) = \frac{1}{19}\left[\frac{11}{13.18}\eta - \frac{11}{3.39}\sin\frac{\eta}{6} + \frac{5}{36}\sin\frac{\eta}{6} - \frac{5}{13.36}\eta^2 - \frac{10}{39}\cos\frac{\eta}{6} + \frac{2}{6}\cos\frac{\eta}{6}\right]$$

We have

$$\varphi(\xi,\eta) = \varphi(0,0) + \xi\varphi_\xi(0,0) + \eta\varphi_\eta(0,0) + \frac{1}{2}\left[\xi^2\varphi_{\xi\xi}(0,0) + 2\xi\eta\varphi_{\xi\eta}(0,0) + \eta^2\varphi_{\eta\eta}(0,0)\right] + \cdots$$

$$\varphi(0,0) = u(0,0) = 0$$

$$\varphi_\xi(0,0) = \frac{1}{39}, \quad \varphi_\eta(0,0) = \frac{1}{6}$$

$$\varphi_{\xi\xi}(0,0) = \frac{1}{19}\left[\frac{-10}{39} + \frac{2}{6}\right] = \frac{1}{13.19}$$

$$\varphi_{\xi\eta}(0,0) = 0, \quad \varphi_{\eta\eta}(0,0) = 0$$

$$\Rightarrow \varphi(\xi,\eta) = \frac{\xi}{39} + \frac{\eta}{6} + \frac{1}{2}\,\frac{1}{13.19}\,\xi^2 + \cdots$$

$$\Rightarrow u(x,y) = \frac{y-5x}{39} + \frac{y+x}{6} + \frac{1}{26.19}(y-5x)^2 + \cdots$$

$$= \frac{1}{26}x + \frac{15}{2.36}y + \frac{1}{26.19}\left(25x^2 - 10xy + y^2\right) + \cdots$$

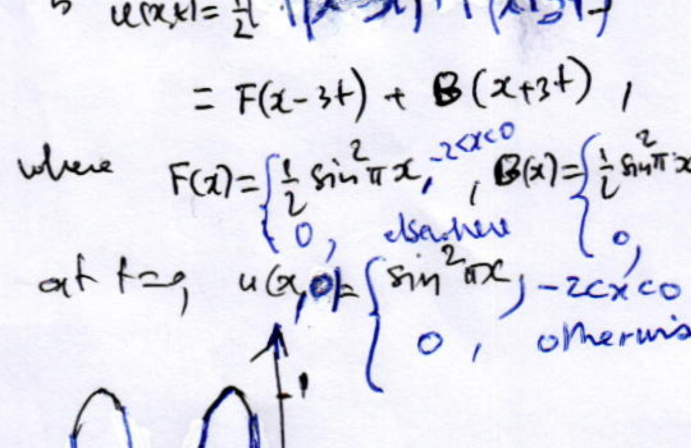
Ex 2: The d'Alembert solution of the wave equation $u_{tt} = 9u_{xx}$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\varphi(x-3t) + \varphi(x+3t)\right]$$

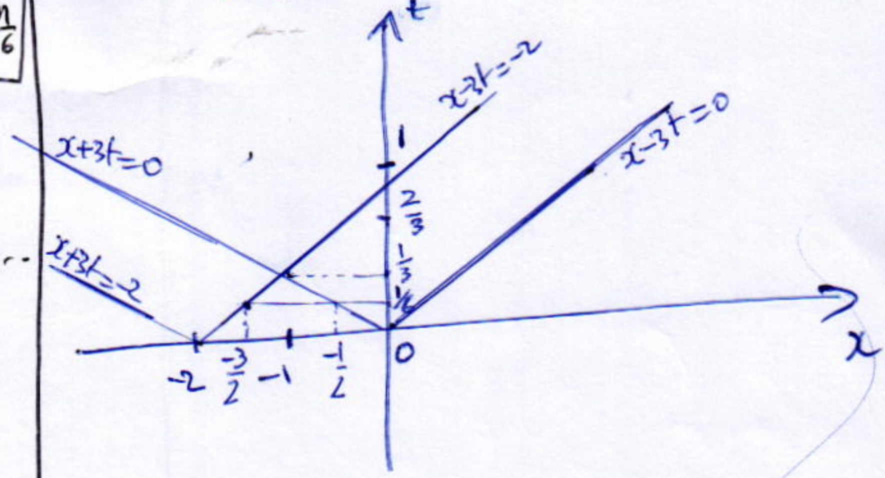
$$= F(x-3t) + B(x+3t),$$

where $F(x) = \begin{cases} \frac{1}{2}\sin^2\pi x, & -2<x<0 \\ 0, & \text{elsewhere} \end{cases}$, $B(x) = \begin{cases} \frac{1}{2}\sin^2\pi x, & -2<x<0 \\ 0, & \text{elsewhere} \end{cases}$

at $t=0$, $u(x,0) = \begin{cases} \sin^2\pi x, & -2<x<0 \\ 0, & \text{otherwise} \end{cases}$



We plot the characteristics curves $x \pm 3t = -2$ and $x \pm 3t = 0$



$\boxed{\text{When } t = 1/6}$

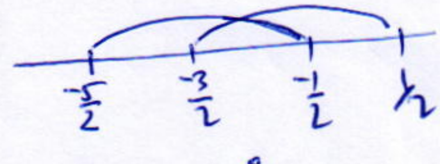
$$u\left(x,\tfrac{1}{6}\right) = \frac{1}{2}\left[\varphi\left(x-\tfrac{1}{2}\right) + \varphi\left(x+\tfrac{1}{2}\right)\right]$$

$$\varphi\left(x-\tfrac{1}{2}\right) = \begin{cases} \sin^2\pi\left(x-\frac{1}{2}\right), & -2 < x-\frac{1}{2} < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\varphi\left(x+\tfrac{1}{2}\right) = \begin{cases} \sin^2\pi\left(x+\frac{1}{2}\right), & -2 < x+\frac{1}{2} < 0 \\ 0, & \text{otherwise} \end{cases}$$

$-2 < x-\frac{1}{2} < 0 \iff -\frac{3}{2} < x < \frac{1}{2}$; $\sin\pi\left(x+\frac{1}{2}\right) = \cos\pi x$

$-2 < x+\frac{1}{2} < 0 \iff -\frac{5}{2} < x < -\frac{1}{2}$; $\sin\pi\left(x-\frac{1}{2}\right) = -\cos\pi x$



$$\Rightarrow u\left(x,\tfrac{1}{6}\right) = \begin{cases} \dfrac{\cos^2\pi x}{2}, & -\frac{5}{2} < x < -\frac{3}{2} \\ \cos^2\pi x, & -\frac{3}{2} < x < -\frac{1}{2} \\ \dfrac{\cos^2\pi x}{2}, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$\boxed{\text{When } t = \frac{1}{3}}$

$$u\left(x,\tfrac{1}{3}\right) = \frac{1}{2}\left[\varphi(x-1) + \varphi(x+1)\right]$$

$$\varphi(x+1)=\begin{cases}\sin^2\pi(x+1), & -2<x+1<0\\ 0, & \text{otherwise}\end{cases}$$

$$\varphi(x-1)=\begin{cases}\sin^2\pi(x-1), & -2<x-1<0\\ 0, & \text{otherwise}\end{cases}$$

$-2<x+1<0 \iff -3<x<-1$

$-2<x-1<0 \iff -1<x<1$

$\sin\pi(x-1) = -\sin\pi x$

$\sin\pi(x+1) = -\sin\pi x$

$$\Rightarrow u(x,\tfrac{1}{3})=\begin{cases}\dfrac{\sin^2\pi x}{2}, & -3<x<-1\\ \dfrac{\sin^2\pi x}{2}, & -1<x<1\\ 0, & \text{otherwise}\end{cases}$$

$\boxed{\text{When } t=2/3}$

$$u(x,\tfrac{2}{3})=\tfrac{1}{2}\left[\varphi(x-2)+\varphi(x+2)\right]$$

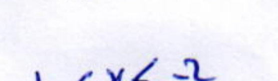
$$\varphi(x-2)=\begin{cases}\sin^2\pi(x-2), & -2<x-2<0\\ 0, & \text{otherwise}\end{cases}$$

$$\varphi(x+2)=\begin{cases}\sin^2\pi(x+2), & -2<x+2<0\\ 0, & \text{otherwise}\end{cases}$$

$-2<x-2<0 \iff 0<x<2$

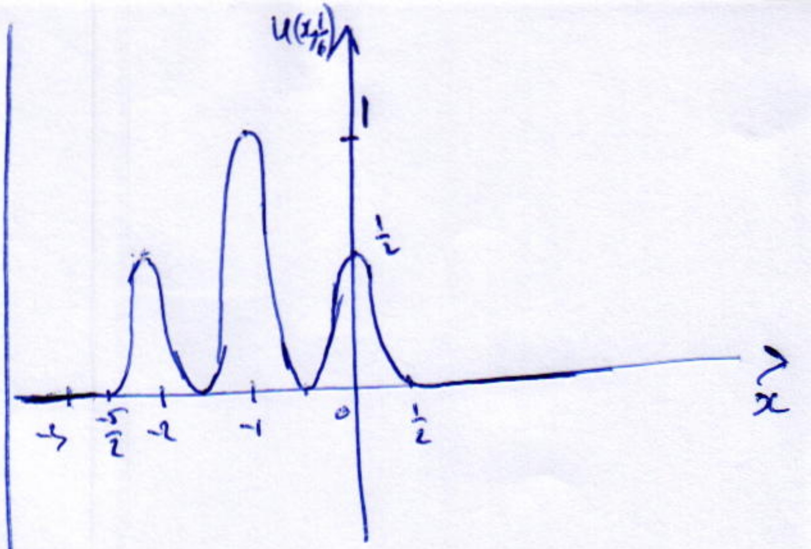
$-2<x+2<0 \iff -4<x<-2$

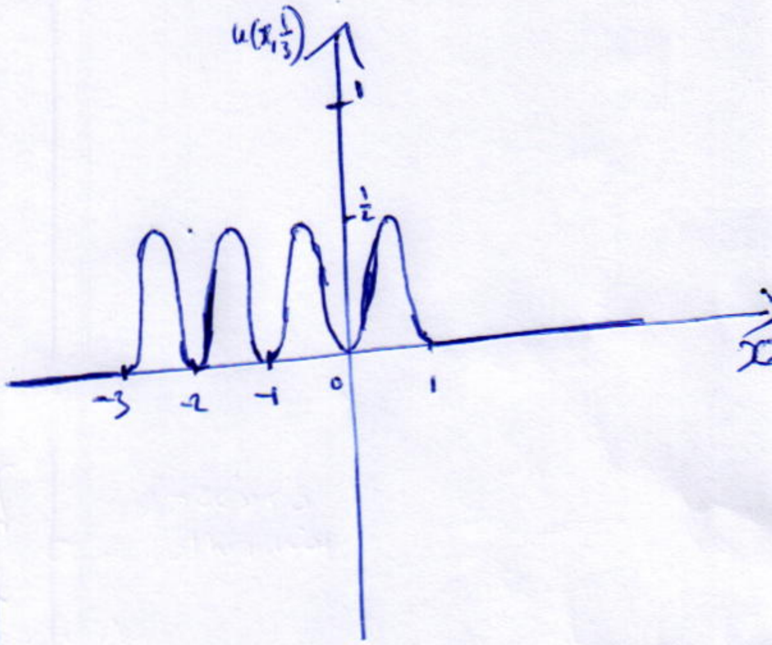
$\sin\pi(x+2) = \sin\pi(x)$

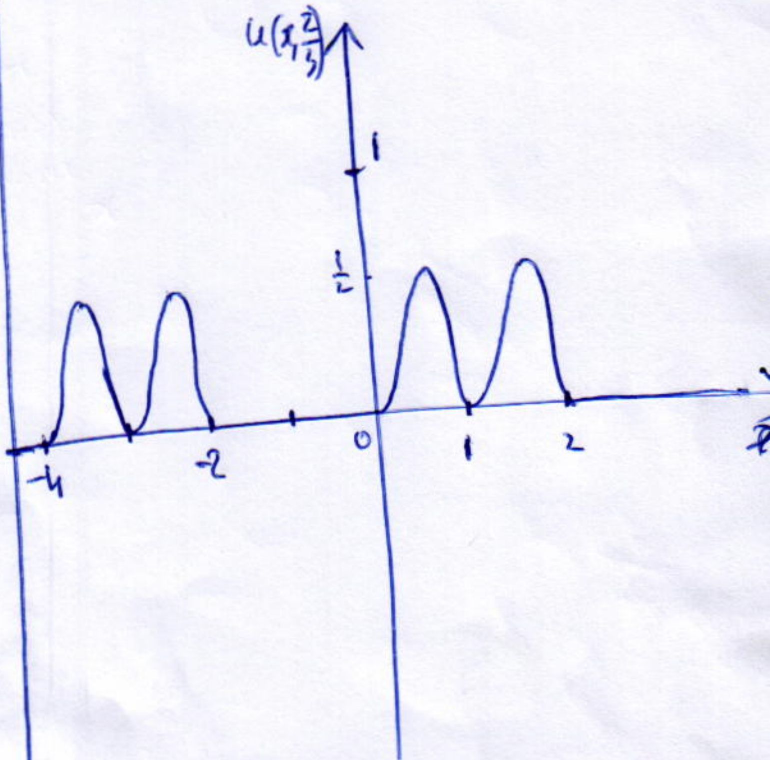


$\sin\pi(x-2) = \sin\pi x$

$$\Rightarrow u(x,\tfrac{2}{3})=\begin{cases}\dfrac{\sin^2\pi x}{2}, & -4<x<-2\\ \dfrac{\sin^2\pi x}{2}, & 0<x<2\\ 0, & \text{otherwise}\end{cases}$$







Ex 3:

$$\begin{cases} u_{tt} = 4u_{xx} & , x>0,\ t>0 \\ u(x,0) = x\sin\pi x \\ u_t(x,0) = x(1-x) \\ u(0,t) = 0 \end{cases}$$

Let $\phi(x) = \begin{cases} x\sin\pi x & , x \geq 0 \\ -x\sin\pi x, & x<0 \end{cases}$

$$\psi(x) = \begin{cases} x(1-x), & x \geq 0 \\ x(1+x), & x<0 \end{cases}$$

ϕ and ψ are odd functions.

We consider the Cauchy problem

$$\begin{cases} u_{tt} = 4u_{xx} & , -\infty < x < \infty \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\phi(x-2t) + \phi(x+2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} \psi(s)\,ds \quad , x>0,\ t>0$$

When $x - 2t \geq 0$,

$$u(x,t) = \frac{1}{2}\left[(x-2t)\sin\pi(x-2t) + (x+2t)\sin\pi(x+2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} s(1-s)\,ds$$

$$= x\sin\pi x\cos 2\pi t + 2t\cos\pi x\sin 2\pi t + t\left(x - x^2 - \frac{4t^2}{3}\right),$$

When $x - 2t < 0$

$$u(x,t) = \frac{1}{2}\left[-(x-2t)\sin\pi(x-2t) + (x+2t)\sin\pi(x+2t)\right] + \frac{1}{4}\int_{x-2t}^{0} s(1+s)\,ds + \frac{1}{4}\int_{0}^{x+2t} s(1-s)\,ds$$

$$= x\cos\pi x\sin 2\pi t + 2t\sin\pi x\cos 2\pi t + tx - \left(\frac{x^3}{6} + 2xt^2\right)$$

$$\Rightarrow u(x,t) = \begin{cases} x\sin\pi x\cos 2\pi t + 2t\cos\pi x\sin 2\pi t + t\left(x - x^2 - \frac{4t^3}{3}\right) \\ \qquad x \geq 2t \geq 0 \\ x\cos\pi x\sin 2\pi t + 2t\sin\pi x\cos 2\pi t \\ \qquad + tx - \left(\frac{x^3}{6} + 2xt^2\right), \quad 0<x<2t \end{cases}$$

$u(x,t)$ is continuous when $x<2t$ and when $x>2t$.

When $x = 2t$, both functions give $4t\sin 2\pi t\sin 2\pi t + 2t^2 - 4t^3 - \frac{4}{3}t^3$

Thus, $u(x,t)$ is continuous on R.

$(R = [0,\infty)\times(0,\infty))$

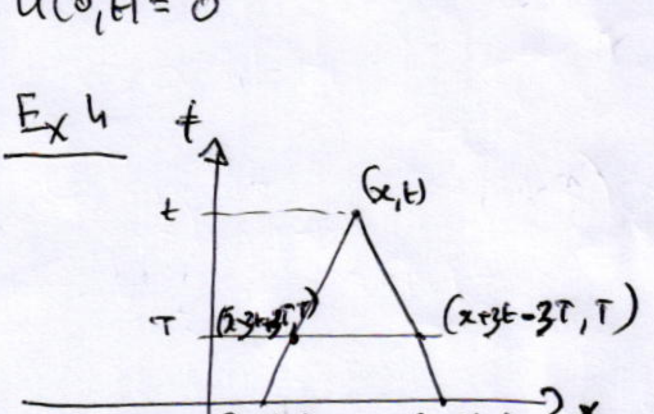
$u(x,0) = x\sin\pi x$

$$u_t(x,t) = \begin{cases} -2\pi x\sin\pi x\sin 2\pi t + 2\cos\pi x\sin 2\pi t \\ \quad + 4t\pi\cos\pi x\cos 2\pi t + x - x^2 - \frac{16}{3}t^3, & x \geq 2t \\ 2\pi x\cos\pi x\cos 2\pi t + 2\sin\pi x\cos 2\pi t + \\ \quad - 4\pi t\sin\pi x\sin 2\pi t + x - 4xt, & 0<x<2t \end{cases}$$

$u_t(x,0) = x - x^2 = x(1-x)$.

$u(0,t) = 0$

Ex 4



$$u(x,t) = \frac{1}{2}\left[3\sin(x-3t) + 3\sin(x+3t)\right] + \frac{1}{6}\int_{x-3t}^{x+3t} s\,d\xi + \frac{1}{6}\iint_{\Delta} 3X T\,dX\,dT$$

But, $\iint_{\Delta} 3xT\,dx\,dT = \int_0^t \left[\int_{x-3t+3T}^{x+3t-3T} 3xT\, dx \right] dT$

$$= \int_0^t \frac{3T}{2}\left[(x+3t-3T)^2 - (x-3t+3T)^2\right] dT$$

$$= \int_0^t 18x\,(t-T)T\,dT$$

$$= 18x\left[\frac{t^3}{2} - \frac{t^3}{3}\right] = 3xt^3.$$

Thus,

$$u(x,t) = \frac{3}{2}\left[\sin(x-3t) + \sin(x+3t)\right] + \frac{1}{12}(12tx) + \frac{1}{2}xt^3$$

$$= \frac{3}{2}\sin(x-3t) + \frac{3}{2}\sin(x+3t) + tx + \frac{1}{2}xt^3.$$

$$u_t = -\frac{9}{2}\cos(x-3t) + \frac{9}{2}\cos(x+3t) + x + \frac{3}{2}xt^2$$

$$u_{tt} = -\frac{27}{2}\sin(x-3t) - \frac{27}{2}\sin(x+3t) + 3xt$$

$$u_x = \frac{3}{2}\cos(x-3t) + \frac{3}{2}\cos(x+3t) + t + 3t^3$$

$$u_{xx} = -\frac{3}{2}\sin(x-3t) - \frac{3}{2}\sin(x+3t)$$

$$\Rightarrow u_{tt} = 9u_{xx} + 3xt.$$

$$u(x,0) = 3\sin x$$

$$u_t(x,0) = x.$$

Ex 5:

$$\begin{cases} u_{tt} + Au_t + Bu = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = 0 \end{cases}$$

$$A^2L^2 < 4(BL^2 + c^2\pi^2)$$

Let $u = XT$

$$\Rightarrow XT'' + AXT' + BXT = c^2X''T$$

$$\frac{T'' + AT' + BT}{c^2T} = \frac{X''}{X} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

$$\begin{cases} T'' + AT' + (B + \lambda c^2)T = 0 \\ T'(0) = 0 \end{cases}$$

$\lambda = 0$ and $\lambda < 0$ give $X(x) = 0$.

- $\lambda > 0$, $\lambda = \alpha^2$

$$X(x) = c_1\cos\alpha x + c_2\sin\alpha x$$

$X(0) = 0 \Rightarrow c_1 = 0$

$X(L) = 0 \Rightarrow \sin\alpha L = 0$, $\alpha L = n\pi$, $\alpha = \frac{n\pi}{L}$

$$X_n(x) = c\sin\frac{n\pi}{L}x.$$

$$T'' + AT' + (B + \alpha_n^2c^2)T = 0$$

$$m^2 + Am + (B + \alpha_n^2c^2) = 0$$

$$\Delta = A^2 - 4(B + \alpha_n^2c^2)$$

$$= A^2 - 4\left(B + \frac{n^2\pi^2c^2}{L^2}\right)$$

$$= \frac{A^2L^2 - 4(BL^2 + n^2\pi^2c^2)}{L^2}$$

Now, $A^2L^2 - 4(BL^2 + n^2\pi^2c^2) < A^2L^2 - 4(BL^2 + \pi^2c^2) < 0$

$$\Rightarrow m = \frac{-A \pm i\sqrt{-\Delta}}{2}$$

$$= -\frac{A}{2} \pm \frac{i}{2L}\sqrt{4(BL^2 + n^2\pi^2c^2) - A^2L^2}$$

$$= -\frac{A}{2} \pm i\beta_n$$

$$T_n(t) = e^{-\frac{At}{2}}\left(c_1\cos\beta_n t + c_2\sin\beta_n t\right)$$

$$T'(t) = \left(-\frac{A}{2}c_1\cos\beta_n t - \frac{A}{2}c_2\sin\beta_n t - c_1\beta_n\sin\beta_n t + c_2\beta_n\cos\beta_n t\right)e^{-\frac{At}{2}}$$

$$T'(0)=0 \Rightarrow -\frac{A}{2}C_1 + C_2\beta_n = 0$$

$$\Rightarrow C_2 = \frac{A}{2\beta_n}C_1$$

$$\Rightarrow T_n(t) = C\left(\cos\beta_n t + \frac{A}{2\beta_n}\sin\beta_n t\right)e^{-\frac{A}{2}t}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} A_n\left(\cos\beta_n t + \frac{A}{2\beta_n}\sin\beta_n t\right)\sin\frac{n\pi}{L}x\, e^{-\frac{A}{2}t}$$

$$u(x,0) = \varphi(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x = \varphi(x)$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L \varphi(x)\sin\frac{n\pi}{L}x\, dx$$