King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 470 Final Exam
The Second Semester of 2012-2013 (122)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		25
3		25
4		30
Total		100

Problem 1: Consider the boundary value problem

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in } D \\ u &= f \quad \text{on } \partial D \end{aligned}$$

where D is a simply-connected 2D region with piecewise smooth boundary ∂D.
1.) State the Maximum Principle for u on D. If $f = 10$ at each point on the boundary ∂D, what is u on D? Explain your answer.
2.) Now let D be the disc of radius R centered at the origin,

$$D = \{(x, y) : \ x^2 + y^2 \leq R\}.$$

Name and state (without proof) another property of u which gives the value of u at the center of the disc in terms of the values of u on the boundary $\partial D = \{(x, y) : \ x^2 + y^2 = R\}$. Use this result to find $u(0, 0)$ if on the boundary u takes the values

$$u(R, \theta) = \begin{cases} 90, & -\pi/2 \leq \theta \leq \pi/2, \\ 25, & \pi/2 \leq \theta \leq \pi, \\ 7, & \pi \leq \theta \leq 3\pi/2. \end{cases}$$

Problem 2: Solve the Laplace equation in the rectangle $0 < x < a$, $0 < y < b$,

$$\nabla^2 v(x, y) = 0,$$

with boundary conditions

$$\begin{aligned} v(0, y) &= v(a, y) = v(x, b) = 0, \\ v(x, 0) &= \cos\left(5\frac{\pi}{a}x\right). \end{aligned}$$

Problem 3: 1.) Use Fourier integral or Fourier transform method to prove that the solution of the Laplace equation for the lower half-plane, whose boundary conditions is the horizontal axis:

$$\begin{aligned} \nabla^2 u(x, y) &= 0, \quad \text{for } -\infty < x < \infty, \ y < 0, \\ u(x, 0) &= f(x) \quad \text{for } -\infty < x < \infty, \end{aligned}$$

is

$$u(x, y) = -\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{y^2 + (\xi - x)^2} d\xi.$$

2.) Write the solution for

$$f(x) = \begin{cases} 0, & |x| > 2, \\ x^2, & -2 \leq x \leq 2. \end{cases}$$

Problem 4: 1.) Solve the Laplace equation on the quarter unit disc

$$\nabla^2 v(r, \theta) = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0,$$

with boundary conditions

$$v(1,\theta) = g(\theta), \quad v(0,\theta) \text{ bounded}, \quad 0 < \theta < \pi/2,$$
$$v(r,0) = 0, \quad v(r,\frac{\pi}{2}) = 0, \quad 0 < r < 1.$$

2.) Solve the heat equation problem on the unit quarter disc

$$u_t = \nabla^2 u(r,\theta,t), \quad 0 < r < 1, \quad 0 < \theta < \pi/2, \quad t > 0,$$

with boundary conditions

$$u(1,\theta,t) = g(\theta), \quad u(0,\theta,t) \text{ bounded}, \quad 0 < \theta < \pi/2, \quad t > 0,$$
$$u(r,0,t) = 0, \quad u(r,\frac{\pi}{2},t) = 0, \quad 0 < r < 1, \quad t > 0,$$

with initial condition

$$u(r,\theta,0) = f(r,\theta), \quad 0 < r < 1, \quad 0 < \theta < \pi/2.$$

Do not evaluate the coefficients in the solution. (Hint: set $w(r,\theta,t) = u(r,\theta,t) - v(r,\theta)$, and solve the problem for w).

3.) Prove the solution to 2.) is unique. (Hint: write the equation for the difference $h = u_1 - u_2$ of two solutions u_1 and u_2, multiply this equation by h, and apply the Divergence Theorem, and do not integrate by parts. No need to use r and θ, just denote the region by D.)

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Q1:

1) u achieves its maximum and minimum values on $\bar{D}$ only at points of ∂D.

let $v = u - 10$.

Thus $\begin{cases} \nabla^2 v = 0 & \text{in } D \\ v = 0 & \text{on } \partial D \end{cases}$

From the Maximum principle,

$v = 0$ in $\bar{D}$

and $u = 10$ on D

2) Mean value property:

$$u(x) = \frac{1}{2\pi\varepsilon} \oint_{\mathcal{C}} u(y)\, d\sigma_y, \text{ for } x \in D$$

$\mathcal{C}$ is a circle of radius ε about x such that interior $\mathcal{C} \subset D$.

Here $\varepsilon = R$, $x = (0,0)$, $D = \text{disc}(0,R)$

$$u(0,0) = \frac{1}{2\pi R} \oint_{\mathcal{C}} u(R,\theta) R\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(\theta)\, d\theta$$

$$= \frac{1}{2\pi}\left(90\pi + 25\frac{\pi}{2} + 7\frac{\pi}{2}\right)$$

$$= \frac{53}{2}$$

Q2:

$$\begin{cases} \nabla^2 v(x,y) = 0 \\ v(0,y) = v(a,y) = v(x,b) = 0 \\ v(x,0) = \cos\left(5\frac{\pi}{a}x\right) \end{cases}$$

Use seperation of variables.

$v = XY$

$X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(a) = 0 \end{cases}, \quad \begin{cases} Y'' - \lambda Y = 0 \\ Y(b) = 0 \end{cases}$$

$\lambda = \alpha^2 \Rightarrow X = C_1 \cos \alpha x + C_2 \sin \alpha x$

$X(0) = 0 \Rightarrow C_1 = 0$

$X(a) = 0 \Rightarrow \sin \alpha a = 0$, $\alpha_n a = n\pi$

$\alpha_n = \frac{n\pi}{a}$, $n = 1, 2, 3 \ldots$

$Y'' - \alpha_n^2 Y = 0$

$Y = C_1 e^{\alpha_n y} + C_2 e^{-\alpha_n y}$

$Y(b) = 0 \Rightarrow C_1 e^{\alpha_n b} + C_2 e^{-\alpha_n b} = 0$

$C_1 = -C_2 e^{-2\alpha_n b}$

$$\Rightarrow Y_n = -C_2 e^{-2\alpha_n b} e^{\alpha_n y} + C_2 e^{-\alpha_n y}$$

$$= -C_2 e^{-\alpha_n b}\left(e^{-\alpha_n b} e^{\alpha_n y} - e^{\alpha_n b} e^{-\alpha_n y}\right)$$

$$= -C_2 e^{-\alpha_n b}\left(e^{+\alpha_n(y-b)} - e^{-\alpha_n(y-b)}\right)$$

$$= C_n \sinh \alpha_n (y-b)$$

$$\Rightarrow v(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}(y-b)$$

$$v(\,,\,) = f(x) = -\sum_{n=1}^{\infty} B_n \sinh\frac{n\pi}{a}b \sin\frac{n\pi}{a}x$$

$$\Rightarrow B_n = -\frac{2}{a\sinh\frac{n\pi}{a}b}\int_0^a f(\xi)\sin\frac{n\pi}{a}\xi\, d\xi$$

$$\int_0^a \cos(5\tfrac{\pi}{a}x)\sin\frac{n\pi}{a}x\,dx =$$

$$\frac{1}{2}\int_0^a\left[\sin(n+5)\frac{\pi}{a}x - \sin(5-n)\frac{\pi}{a}x\right]dx$$

$$= \frac{1}{2}\left[\frac{-\cos(n+5)\frac{\pi}{a}x}{(n+5)\frac{\pi}{a}} + \frac{\cos(5-n)\frac{\pi}{a}x}{(5-n)\frac{\pi}{a}}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{(-1)^n+1}{(n+5)\frac{\pi}{a}} + \frac{-(-1)^n-1}{(5-n)\frac{\pi}{a}}\right]$$

$$= \begin{cases} 0 & \text{if } n=2k+1 \\ \frac{1}{2}\left(\frac{2}{(2k+5)}\frac{a}{\pi} - \frac{2}{(5-2k)}\frac{a}{\pi}\right) & n=2k \end{cases}$$

$$= \begin{cases} 0, & \text{if } n=2k+1 \\ \frac{4k}{4k^2-25}\frac{\pi}{a}, & \text{if } n=2k \end{cases}$$

$$\boxed{v(x,y) = \frac{-\pi}{a^2}\frac{8}{\sinh\frac{n\pi}{a}b}\sum_{k}^{\infty}\frac{k}{4k^2-25}\sin\frac{2k\pi}{a}x\,\sinh\frac{2k\pi}{a}(y-b)}$$

Q:3

Fourier integral method

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

- $\lambda = 0$, $X = c_1 x + c_2$

  X bounded $\Rightarrow c_1 = 0$

  $Y'' = 0$, $Y = c_1 y + c_2$

  Y bounded $\Rightarrow c_1 = 0$

  $\Rightarrow u(x,y) = C$

- $\lambda = -w^2$, $X(x) = c_1 e^{wx} + c_2 e^{-wx}$

  X bounded $\Rightarrow c_1 = c_2 = 0$

  Thus $u(x,y) = 0$

- $\lambda = w^2$

  $X(x) = c_1\cos wx + c_2 \sin wx$

  $Y(y) = c_1 e^{wy} + c_2 e^{-wy}$, $y<0$

  Y bounded when $y \to -\infty \Rightarrow c_2 = 0$

  $Y(y) = c\, e^{wy}$

$$\Rightarrow u(x,y) = \int_0^{\infty}(a_w \cos wx + b_w \sin wx)\, e^{wy}\, dw$$

$$a_w = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos w\xi\, d\xi$$

$$b_w = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin w\xi\, d\xi$$

$$\Rightarrow u(x,y) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}(\cos w\xi\cos wx + \sin w\xi \sin wx) f(\xi) e^{wy}\, d\xi\, dw$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\int_0^{\infty}\cos w(\xi - x)\, e^{wy}\, dw\right) f(\xi)\, d\xi$$

$$= -\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\cdot}{y^2+(\xi-x)^2} f(\xi)\, d\xi$$

Method of Fourier transform

Let $\mathcal{F}\{u(x,y)\} = \hat{u}(\omega,y)$

$$= \int_{-\infty}^{\infty} u(x,y)\, e^{-i\omega x} dx$$

$$\mathcal{F}\{u_{xx}(x,y)\} = -\omega^2 \hat{u}(\omega,y).$$

Taking Fourier transform of the equation, we find

$$\frac{\partial^2}{\partial x^2}\hat{u} - \omega^2 \hat{u} = 0$$

$$\Rightarrow \hat{u} = a_\omega e^{\omega y} + b_\omega e^{-\omega y}, \quad y < 0$$

when $\omega > 0$, $e^{-\omega y} \to \infty \Rightarrow b_\omega = 0$

when $\omega < 0$, $e^{\omega y} \to \infty \Rightarrow a_\omega = 0$

$$\Rightarrow \hat{u} = c(\omega) e^{|\omega| y}$$

$$\hat{u}(\omega,0) = \hat{f}(\omega) \Rightarrow c(\omega) = \hat{f}(\omega)$$

$$\Rightarrow \hat{u}(\omega,y) = \hat{f}(\omega)\, e^{|\omega| y}$$

$$\Rightarrow u(x,y) = \mathcal{F}^{-1}\{\hat{u}(\omega,y)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{|\omega| y} e^{-i\omega x} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{|\omega| y} e^{+i\omega\xi} f(\xi)\, d\xi\right) e^{-i\omega x} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{|\omega| y} e^{-i\omega(\xi - x)} d\omega\right) f(\xi)\, d\xi$$

$$= \frac{-y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{y^2 + (\xi - x)^2}\, d\xi .$$

2.)
$$u(x,y) = -y \int_{-2}^{2} \frac{\xi^2}{y^2 + (\xi - x)^2}\, d\xi$$

Let $\tau = \xi - x$,

$$\Rightarrow d\tau = d\xi$$

$$u(x,y) = -y \int_{-2-x}^{2-x} \frac{(\tau + x)^2}{y^2 + \tau^2}\, d\tau$$

$$= -y \int_{-2-x}^{2-x} \left(\frac{\tau^2}{y^2+\tau^2} + \frac{2x\tau}{y^2+\tau^2} + \frac{x^2}{y^2+\tau^2}\right) d\tau$$

$$= -y \int_{-2-x}^{2-x} \left(1 - \frac{y^2}{y^2+\tau^2} + \frac{2x\tau}{y^2+\tau^2} + \frac{x^2}{y^2+\tau^2}\right) d\tau$$

$$= -y \int_{-2-x}^{2-x} \left(1 + 2x\frac{\tau}{y^2+\tau^2} + \frac{x^2 - y^2}{y^2+\tau^2}\right) d\tau$$

$$= -y \left[\tau + x\ln(y^2+\tau^2) + \frac{(x^2-y^2)}{y}\tan^{-1}\left(\frac{\tau}{y}\right)\right]_{-2-x}^{2-x}$$

$$= -4y - xy\ln\left(\frac{y^2+(2-x)^2}{y^2+(2+x)^2}\right) - (x^2-y^2)\left(\tan^{-1}\left(\frac{2-x}{y}\right) + \tan^{-1}\left(\frac{2+x}{y}\right)\right)$$

Q 4

In polar coordinates, the Laplace equation is

$$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0$$

$$v = R\Theta$$

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0$$

$$r^2\frac{\left(R'' + \frac{1}{r}R'\right)}{R} = -\frac{\Theta''}{\Theta} = -\lambda$$

$$\begin{cases} r^2R'' + rR' + \lambda R = 0 \end{cases}$$

and $\begin{cases} \Theta'' - \lambda\Theta = 0 \\ \Theta(0) = \Theta(\frac{\pi}{2}) = 0 \end{cases}$

• $\lambda = 0 \Rightarrow \quad \Theta = c_1\theta + c_2$

$\Theta(0) = 0 \Rightarrow c_2 = 0$

$\Theta(\frac{\pi}{2}) = 0 \Rightarrow c_1 = 0$

Thus $\Theta = 0$

• $\lambda = \alpha^2$, $\Theta = c_1\cosh\alpha\theta + c_2\sinh\alpha\theta$

$\Theta(0) = 0 \Rightarrow c_1 = 0$

$\Theta(\frac{\pi}{2}) = 0 \Rightarrow c_2 = 0$

Thus, $\Theta = 0$

• $\lambda = -\alpha^2$, $\Theta(\theta) = c_1\cos\alpha\theta + c_2\sin\alpha\theta$

$\Theta(0) = 0 \Rightarrow c_1 = 0$

$\Theta(\frac{\pi}{2}) = 0 \Rightarrow \sin\alpha\frac{\pi}{2} = 0$

$\frac{\alpha\pi}{2} = n\pi$, $n = 1, 2, 3, \ldots$

$\alpha_n = 2n$

$\Rightarrow \Theta_n(\theta) = c_n\sin 2n\theta$

Now, we solve

$r^2R'' + rR' - (2n)^2R = 0$

$\Rightarrow R(r) = c_1r^{2n} + c_2r^{-2n}$

R must be bounded at $r=0$

$\Rightarrow \quad c_2 = 0$ and $R_n(r) = c_nr^{2n}$.

$$\Rightarrow v(r,\theta) = \sum_{n=1}^{\infty} B_n\sin 2n\theta\, r^{2n}$$

$v(1,\theta) = g(\theta) \Rightarrow$

$$\sum_{n=1}^{\infty} B_n\sin 2n\theta = g(\theta)$$

$$\Rightarrow B_n = \frac{4}{\pi}\int_0^{\pi/2}\sin 2n\varphi\, g(\varphi)\,d\varphi$$

2) let $w(r,\theta,t) = u(r,\theta,t) - v(r,\theta)$.

w satisfies the problem

$$\begin{cases} w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} = w_t \\ w(1,\theta,t) = 0 \\ w(r,0,t) = 0, \quad w(r,\frac{\pi}{2},t) = 0 \end{cases}$$

$w = TR\Theta$

$T'R\Theta = TR''\Theta + \frac{1}{r}R'T\Theta + \frac{1}{r^2}R\Theta''T$

$$\frac{T'}{T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda$$

$\begin{cases} T' + \lambda T = 0 \end{cases}$, $r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} + r^2\lambda = 0$

$$\frac{\Theta''}{\Theta} = -\left(r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda\right) = -\mu$$

$\begin{cases} \Theta'' + \mu\Theta = 0 \\ \Theta(0) = \Theta(\frac{\pi}{2}) = 0 \end{cases}$, $\begin{cases} r^2R'' + rR' + (r^2\lambda - \mu)R = 0 \\ R(1) = 0 \end{cases}$

$\mu_n = (2n)^2$, $n = 1, 2, \ldots$

$\Theta_n(\theta) = c_n\sin 2n\theta$

$T' + \lambda T = 0 \Rightarrow T(t) = ce^{-\lambda t}$, $\underline{\lambda \geq 0}$

$$r^2R'' + rR' + (r^2\lambda - (2n)^2)R = 0$$

$$\Rightarrow R(r) = c_1 J_{2n}(\sqrt{\lambda}r) + c_2 Y_{2n}(\sqrt{\lambda}r)$$

$R(0)$ must be bounded

$\Rightarrow c_2 = 0$

$R(1) = 0 \Rightarrow J_{2n}(\sqrt{\lambda}) = 0$

$\sqrt{\lambda_{mn}} = \alpha_{mn}, \quad \lambda_{mn} = \alpha_{mn}^2$

$$\Rightarrow w(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} J_{2n}(\alpha_{nm}r)\sin 2n\theta\, e^{-\lambda_{nm}t}$$

3.) Assume there are two solutions u_1 and u_2.

Set $h = u_1 - u_2$

$$\begin{cases} h_t = \nabla^2 h \\ h(1,\theta,t) = 0, \quad h(0,\theta,t) \text{ bounded} \\ h(r,0,t) = h(r,\frac{\pi}{2},t) = 0 \\ h(r,\theta,0) = 0 \end{cases}$$

We multiply by h, we over D, we find

$$\frac{1}{2}\frac{d}{dt}\iint_D h^2\,dA = \iint_D h\nabla^2 h\,dA$$

The divergence theorem says $\int_{\partial D} v\nabla v\cdot n\,ds = \iint_D |\nabla v|^2 dA + \iint_D v\nabla^2 v\,dA$

$\Rightarrow$

$$\frac{1}{2}\frac{d}{dt}\iint_D h^2\,dA + \iint_D |\nabla h|^2 dA = \int_{\partial D} h\nabla h\cdot n\,ds$$

$h = 0$ on ∂D

$$\Rightarrow \frac{d}{dt}\iint_D h^2\,dA \le 0$$

$$\iint_D h^2\,dA \le h^2(r,\theta,0) = 0$$

$$\Rightarrow h^2(r,\theta,t) = 0$$

$\Rightarrow h = 0$

and $u_1 = u_2$