King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 470 Major Exam 2
The Second Semester of 2012-2013 (122)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		22
2		22
3		20
4		18
5		18
Total		100

Problem 1: Suppose that an infinite string has initial displacement

$$u(x,0) = f(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-2x, & 0 \le x \le 1/2 \\ 0, & x < -1 \text{ and } x > 1/2 \end{cases}$$

and zero initial velocity $u_t(x,0) = 0$. Write down the solution of the wave equation

$$u_{tt} = u_{xx}$$

with ICs $u(x,0) = f(x)$ and $u_t(x,0) = 0$ using D'Alembert's formula. Illustrate the nature of the solution by sketching the ux-profiles $y = u(x,t)$ of the string displacement for $t = 0$, $t = 1/2$ and $t = 1$.

Problem 2: The acoustic pressure in an organ pipe obeys the 1-D wave equation

$$p_{tt} = c^2 p_{xx}$$

where c is the speed of sound in air. Each organ pipe is closed at one end and open at the other. At the closed end, the BC is that $p_x(0,t) = 0$, while at the open end, the BC is $p(L,t) = 0$, where L is the length of the pipe.Given initial conditions $p(x,0) = f(x)$ and $p_t(x,0) = g(x)$, use separation of variables to determine the Fourier-series solution of the IBVP.

Problem 3: Solve the IBVP

$$\begin{aligned} &u_{tt} = 9u_{xx} + 54x^2 \quad \text{for } 0 < x < 1,\ t > 0 \\ &u(x,0) = u_t(x,0) = 0 \quad \text{for } 0 \le x \le 1 \\ &u(0,t) = u(1,t) = 0 \quad \text{for}\, t \ge 0. \end{aligned}$$

(Hint: write $u(x,t) = U(x,t) + f(x)$ in which U satisfies an homogeneous PDE and solve this PDE to find U, then deduce the solution u)

Problem 4: Consider the homogeneous heat problem

$$u_t = u_{xx}; \quad u_x(0,t) = 0 = u_x(1,t); \quad u(x,0) = f(x)$$

where $t > 0$, $0 \le x \le 1$ and f is a piecewise smooth function on $[0,1]$.
1.) Determine the Fourier-solution of the IBVP.
2.) Write explicitly the solution in the case $f(x) = u_0$, u_0 is a constant.

Problem 5: Consider the diffusion problem on the positive spatial axis

$$u_t = ku_{xx}, \quad \text{for } x > 0,\ t > 0,$$

with initial and boundary conditions

$$u(x,0) = f(x) \ \text{ for } x \ge 0, \quad u_x(0,t) = 0 \ \text{ for } t \ge 0.$$

Here k is a positive constant, f a given function. Apply separation of variables to obtain a bounded solution of the problem in terms of Fourier integral.

MATH 470 (122)
Exam 2

1.) $u(x,t)=\frac{1}{2}\left[f(x-t)+f(x+t)\right]$

when $t=0$,

$$u(x,0)=f(x)=\begin{cases} x+1, & -1\le x\le 0\\ 1-2x, & 0\le x\le \frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

when $t=\frac{1}{2}$

$$u(x,\tfrac{1}{2})=\tfrac{1}{2}\left[f(x-\tfrac{1}{2})+f(x+\tfrac{1}{2})\right]$$

$$f(x-\tfrac{1}{2})=\begin{cases} x-\frac{1}{2}+1, & -1\le x-\frac{1}{2}\le 0\\ 1-2(x-\frac{1}{2}) & 0\le x-\frac{1}{2}\le \frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

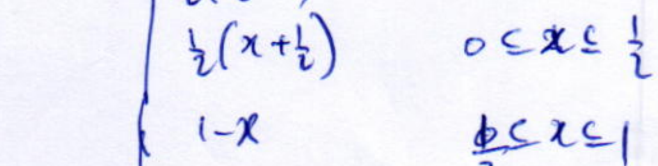
$$f(x+\tfrac{1}{2})=\begin{cases} x+\frac{1}{2}+1, & -1\le x+\frac{1}{2}\le 0\\ 1-2(x+\frac{1}{2}), & 0\le x+\frac{1}{2}\le \frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

$-1\le x-\frac{1}{2}\le 0 \iff -\frac{1}{2}\le x\le \frac{1}{2}$

$0\le x-\frac{1}{2}\le \frac{1}{2} \iff \frac{1}{2}\le x\le 1$

$-1\le x+\frac{1}{2}\le 0 \iff -\frac{3}{2}\le x\le -\frac{1}{2}$

$0\le x+\frac{1}{2}\le \frac{1}{2} \iff -\frac{1}{2}\le x\le 0$



$$\Rightarrow u(x,\tfrac{1}{2})=\begin{cases} \frac{1}{2}(x+\frac{3}{2}), & -\frac{3}{2}\le x\le -\frac{1}{2}\\ \frac{1}{2}(\frac{1}{2}-x), & -\frac{1}{2}\le x\le 0\\ \frac{1}{2}(x+\frac{1}{2}) & 0\le x\le \frac{1}{2}\\ 1-x & \frac{1}{2}\le x\le 1\\ 0 & \text{otherwise}\end{cases}$$

when $t=1$

$$u(x,1)=\tfrac{1}{2}\left[f(x-1)+f(x+1)\right]$$

$$f(x-1)=\begin{cases} x-1+1, & -1\le x-1\le 0\\ 1-2(x-1), & 0\le x-1\le \frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

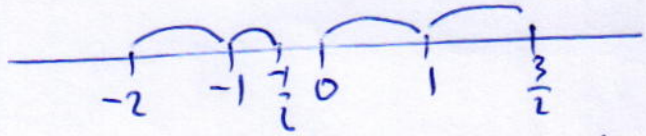
$$f(x+1)=\begin{cases} x+1+1, & -1\le x+1\le 0\\ 1-2(x+1), & 0\le x+1\le \frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

$-1\le x-1\le 0 \iff 0\le x\le 1$

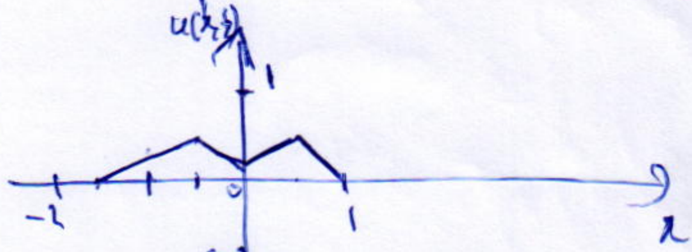
$0\le x-1\le \frac{1}{2} \iff 1\le x\le \frac{3}{2}$

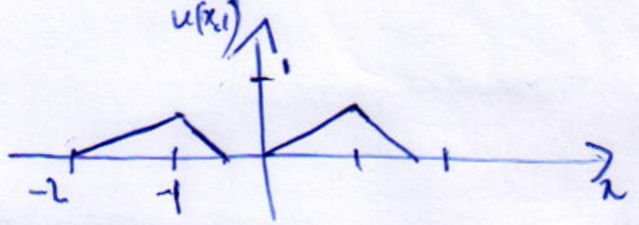
$-1\le x+1\le 0 \iff -2\le x\le -1$

$0\le x+1\le \frac{1}{2} \iff -1\le x\le -\frac{1}{2}$



$$u(x,1)=\begin{cases} \frac{1}{2}(x+2), & -2\le x\le -1\\ -\frac{1}{2}(1+2x), & -1\le x\le -\frac{1}{2}\\ \frac{x}{2}, & 0\le x\le 1\\ \frac{1}{2}(3-2x), & 1\le x\le \frac{3}{2}\\ 0, & \text{otherwise}\end{cases}$$





[2].

$$\begin{cases} p_{tt} = c^2 p_{xx}, & 0<x<L,\ t>0 \\ p_x(0,t) = 0 = p(L,t) = 0 \\ p(x,0) = f(x), & p_t(x,0) = g(x) \end{cases}$$

Let $p(x,t) = X(x)T(t)$

$$XT'' = c^2 X'' T \Leftrightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

$$\Leftrightarrow \begin{cases} X'' + \lambda X = 0 \\ X'(0) = X(L) = 0 \end{cases} \quad (1)$$

$$T'' + c^2\lambda T = 0$$

We solve (1)

- $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = c_1 + c_2 x$

$X'(x) = c_2$, $X'(0) = 0 \Rightarrow c_2 = 0$

$X(L) = 0 \Rightarrow c_1 = 0$

Thus, $p = 0$

- $\lambda < 0$, $\lambda = -\alpha^2$

$X(x) = c_1 \cdot \cosh\alpha x + c_2 \sinh\alpha x$

$X'(x) = c_1\alpha \sinh\alpha x + c_2 \alpha \cosh\alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X(L) = 0 \Rightarrow c_1 \cosh\alpha L = 0$

$\Rightarrow c_1 = 0$

Thus, $p = 0$

- $\lambda > 0$, $\lambda = \alpha^2$

$X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$

$X'(x) = -c_1\alpha \sin\alpha x + c_2 \alpha \cos\alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X(L) = 0 \Rightarrow c_1 \cos\alpha L = 0$

$\cos\alpha L = 0$

$\alpha L = \frac{\pi}{2} + n\pi$

$\Rightarrow \alpha_n = \left(\frac{1}{2} + n\right)\frac{\pi}{L} = (2n+1)\frac{\pi}{2L}$

$n = 0, 1, 2, \ldots$

$X_n(x) = c_n \cos\left(\frac{2n+1}{2}\frac{\pi}{L}\right)$

$T'' + c^2\alpha_n^2 T = 0$

$T_n(t) = c_1 \cos(c\alpha_n t) + c_2 \sin(c\alpha_n t)$

$$\Rightarrow p(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos(c\alpha_n t) + B_n \sin(c\alpha_n t)\right) \cos\alpha_n x$$

$$p_t(x,t) = \sum_{n=0}^{\infty} c\alpha_n \left(-A_n \sin c\alpha_n t + B_n \cos c\alpha_n t\right) \cos\alpha_n x$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} A_n \cos\alpha_n x$$

$$\Rightarrow A_n = \frac{\int_0^L f(x) \cos\left(\frac{(2n+1)\pi}{2L}x\right) d}{\int_0^L \cos^2\left(\frac{(2n+1)\pi}{2L}x\right) dx}$$

$$= \frac{2}{L}\int_0^L f(x) \cos\left(\frac{(2n+1)}{2}\frac{\pi}{L}x\right) dx$$

$$p_t(x,0) = g(x) = \sum_{n=0}^{\infty} c\alpha_n B_n \cos\alpha_n x$$

$$c\alpha_n B_n = \frac{2}{L}\int_0^L g(x) \cos\alpha_n x \, dx$$

$$B_n = \frac{2}{cL\alpha_n}\int_0^L g(x) \cos\alpha_n x \, dx$$

$\boxed{3}$.

$$u(x,t) = U(x,t) + f(x)$$

$$u_{tt} = U_{tt}$$

$$u_{xx} = U_{xx} + f''(x)$$

$$u_{tt} = 9u_{xx} + 54x^2 \iff$$

$$U_{tt} = 9\left(U_{xx} + f''(x)\right) + 54x^2$$
$$= 9U_{xx} + \underbrace{9f''(x) + 54x^2}_{=0}$$

$$u(x,0) = 0 \Rightarrow U(x,0) + f(x) = 0$$

$$u_t(x,0) = 0 \Rightarrow U_t(x,0) = 0$$

$$u(0,t) = 0 \Rightarrow U(0,t) + f(0) = 0$$

$$u(1,t) = 0 \Rightarrow U(1,t) + f(1) = 0$$

Thus, we have

$$\begin{cases} U_{tt} = 9U_{xx} \\ U(x,0) = -f(x) \\ U_t(x,0) = 0 \\ U(0,t) = U(1,t) = 0 \end{cases} \quad \underline{\text{and}} \quad \begin{cases} 9f''(x) + 54x^2 = 0 \\ f(0) = f(1) = 0 \end{cases}$$

We find $f(x)$.

$$f''(x) = -6x^2$$

$$f'(x) = -2x^3 + c_1$$

$$\Rightarrow f(x) = -\frac{1}{2}x^4 + c_1 x + c_2$$

$$f(0) = 0 \Rightarrow c_2 = 0$$

$$f(1) = 0 \Rightarrow c_1 = \frac{1}{2}$$

$$f(x) = -\frac{1}{2}x^4 + \frac{1}{2}x$$

Now, it is easy to solve the IBVP in U.

$$\Rightarrow U(x,t) = XT$$

$$9X''T = XT'' \iff \frac{T''}{9T} = \frac{X''}{X} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(1) = 0 \end{cases}$$

$$\begin{cases} T'' + 9\lambda T = 0 \\ T'(0) = 0 \end{cases}$$

$\lambda = \alpha^2$, $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

$$X(0) = 0 \Rightarrow c_1 = 0$$

$$X(1) = 0 \Rightarrow \sin\alpha = 0,\ \alpha = n\pi$$

$$X_n(x) = c_n \sin n\pi x$$

$$T_n(t) = c_1 \cos 3n\pi t + c_2 \sin 3n\pi t$$

$$T_n'(t) = -3n\pi c_1 \sin 3n\pi t + 3n\pi c_2 \cos 3n\pi t$$

$$T_n'(0) = 0 \Rightarrow c_2 = 0$$

$$\Rightarrow U(x,t) = \sum_{n=1}^{\infty} A_n \cos 3n\pi t \sin n\pi x$$

$$-f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

$$A_n = -2\int_0^1 f(x) \sin n\pi x \, dx$$

$$u(x,t) = U(x,t) - \frac{1}{2}x^4 + \frac{1}{2}x.$$

$\boxed{Ex. 4}$

$$\begin{cases} u_t = u_{xx} \\ u_x(0,t) = 0 = u_x(1,t) \\ u(x,0) = f(x) \end{cases}$$

$u(x,t) = X(x)\,T(t)$

$X T' = X'' T$

$\frac{X''}{X} = \frac{T'}{T} = -\lambda$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(1) = 0 \end{cases} \quad (1)$$

$T' + \lambda T = 0$

We solve (1)

- $\lambda = 0$, $X'' = 0$, $X(x) = c_1 + c_2 x$

  $X'(x) = c_2 \Rightarrow c_2 = 0$

  $\Rightarrow X(x) = c_1$

  $T' = 0 \Rightarrow T(t) = c$

  $\Rightarrow u(x,t) = C$

- $\lambda < 0$, $\lambda = -\alpha^2$

  $X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

  $X'(x) = c_1 \alpha \sinh \alpha x + c_2 \alpha \cosh \alpha x$

  $X'(0) = 0 \Rightarrow c_2 = 0$

  $X'(1) = 0 \Rightarrow c_1 = 0$

  $\Rightarrow X(x) = 0$

- $\lambda > 0$, $\lambda = \alpha^2$

  $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

  $X'(x) = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$

  $X'(0) = 0 \Rightarrow c_2 = 0$

  $X'(1) = 0 \Rightarrow \sin \alpha = 0$, $\alpha_n = n\pi$

  $\Rightarrow X_n(x) = c_n \cos n\pi x$, $n = 1, 2, ..$

$T' + \alpha_n^2 T = 0$

$T_n(t) = c_n e^{-\alpha_n^2 t} = c_n e^{-n^2\pi^2 t}$

$\Rightarrow u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\pi x \, e^{-n^2\pi^2 t}$

at $t=0$, $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\pi x$

$\Rightarrow \frac{A_0}{2} = \int_0^1 f(x)\,dx \Rightarrow A_0 = 2\int_0^1 f(x)\,dx$

$A_n = 2\int_0^1 f(x) \cos n\pi x \, dx$.

2.) When $f(x) = u_0$

$A_0 = 2\int_0^1 u_0 \, dx = 2u_0$

$A_n = 2\int_0^1 u_0 \cos n\pi x \, dx = 0$

$\Rightarrow u(x,t) = u_0$.

$\boxed{5}$ $\begin{cases} u_t = k u_{xx} \,, & x > 0 \\ u(x,0) = f(x) \\ u_x(0,t) = 0 \end{cases}$

$u(x,t) = X(x)T(t)$

$XT' = kX''T$

$\dfrac{T'}{kT} = \dfrac{X''}{X} = -\lambda$

$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \end{cases}$

$T' + \lambda k T = 0$

• $\lambda = 0$, $X'' = 0$, $X(x) = c_1 + c_2 x$

$X'(x) = c_2 \Rightarrow c_2 = 0$

$\Rightarrow X(x) = c$

$T(t) = c \Rightarrow u(x,t) = c$

• $\lambda < 0$, $\lambda = -\alpha^2$

$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$X'(x) = \alpha c_1 e^{\alpha x} - c_2 \alpha e^{-\alpha x}$

$X'(0) = 0 \Rightarrow c_2 = c_1$

$X(x)$ bounded $\Rightarrow c_1 = 0$

Thus, $u = 0$

• $\lambda > 0$, $\lambda = \alpha^2$

$X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$

$X'(x) = -c_1 \alpha \sin\alpha x + c_2 \alpha \cos\alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X(x) = c \cos\alpha x$

$T' + \alpha^2 k T = 0 \Rightarrow T(t) = c e^{-\alpha^2 k t}$

$\Rightarrow u(x,t) = \int_0^\infty A_\alpha \cos\alpha x \, e^{-\alpha^2 k t} \, d\alpha$

$f(x) = \int_0^\infty A_\alpha \cos\alpha x \, d\alpha$

$A_\alpha = \dfrac{2}{\pi} \int_0^\infty f(\xi) \cos\alpha\xi \, d\xi$

End