King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 470 Major Exam 1
The Second Semester of 2012-2013 (122)

Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		18
2		23
3		23
4		18
5		18
Total		100

Problem 1: Consider the first order partial differential equation

$$u_x + e^x u_y = 1. \tag{1}$$

1.) Write down the characteristic equation, and determine an explicit expression for the characteristic curve in the x-y-plane.
2.) Apply a transformation induced by the characteristics, and try to find a general solution of the transformed equation. After transforming that result back to the original variables, verify explicitly that your solution satisfies equation (1).

Problem 2: Consider the first order quasi-linear partial differential equation

$$u_x + e^x u_y = 1. \tag{2}$$

1.) Determine the characteristics of (1), here meant as curves in x-y-u-space.
2.) Solve the equation (2) around the x-axis, for the Cauchy data $u(x, 0) = 1$ given on the x-axis.
3.) Find two different solutions of equation (2) for Cauchy data $u(x, e^x) = x$ on the curve $y = e^x$.
4.) Show that there is no solution of equation (2) for Cauchy data $u(x, e^x) = 1$ (here it is sufficient to use the expression for the Cauchy data and the general solution of (2) found in Problem 1).

Problem 3: Consider the linear second order PDE with parameter α

$$u_{xx} + \alpha u_{xy} + 7u_{yy} = 0. \tag{3}$$

1.) Depending on the value of α, determine the type of the PDE (hyperbolic, parabolic, or elliptic).
2.) For $\alpha = -8$,
a.) State the characteristics equations of (3).
b.) Solve these for the characteristics curves, and sketch some of the characteristics.
c.) Apply a change of variables and transform the PDE (3) to its canonical form.
d.) Find a general solution of the transformed equation.
e.) Transform the solution back to the original variables x and y.
f.) Verify that the general solution you've found satisfies the original equation (3).

Problem 4: Consider the linear second order PDE with parameter α

$$u_{xx} + \sqrt{28}u_{xy} + 7u_{yy} = 0. \tag{4}$$

1.) State the characteristics equations of (4).
2.) Solve these for the characteristics curves.
3.) Apply a change of variables and transform the PDE (4) to its canonical form.
4.) Find a general solution of the transformed equation.
5.) Transform the solution back to the original variables x and y.
6.) Verify that the general solution you've found satisfies the original equation (4).

Problem 5: Consider the Cauchy problem for second order PDE

$$u_{xx} - 4u_{xy} + yu_{yy} + u_x + yu_y = 0 \tag{5}$$

with Cauchy data $u(0, y) = y^3$ and $u_x(0, y) = 4y$ given on the y-axis.
1.) Verify that the y-axis is not a characteristic of (5).
2.) Use the Cauchy data and the PDE to compute the first four terms of the Taylor expansion of the solution of the Cauchy problem around the y-axis.

Exam 1 (MATH 470, term 122)

1) $u_x + e^x u_y = 1$

a) $\frac{dy}{dx} = e^x$ is the characteristic equation. We integrate this equation, and we find

$y = e^x + C$.

Let $\eta = y - e^x$.

We now choose $\xi = x$.

$J = \begin{vmatrix} 1 & 0 \\ -e^x & 1 \end{vmatrix} = 1 \neq 0$

b) $u_x = w_\xi \xi_x + w_\eta \eta_x$

$= w_\xi - e^x w_\eta$

$u_y = w_\xi \xi_y + w_\eta \eta_y = w_\eta$

$\Rightarrow w_\xi - e^x w_\eta + e^x (w_\eta) = 1$

$\Rightarrow w_\xi = 1$

$\Rightarrow w(\xi, \eta) = \xi + f(\eta)$

So that, $\boxed{u(x,y) = x + f(y - e^x)}$ where f is an arbitrary differentiable function.

$u_x = 1 - e^x f'(y - e^x)$

$u_y = f'(y - e^x)$

$u_x + e^x u_y = 1 - e^x f'(y - e^x) + e^x f'(y - e^x)$

$= 1.$

$\Rightarrow u(x,y)$ is a solution to the initial PDE.

b) Method of characteristics

i) $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = e^x$, $\frac{du}{dt} = 1$

$x = t + a$, $\frac{dy}{dx} = e^x \Rightarrow y = e^x + b$

$\Rightarrow y = e^{t+a} + b$

and $u = t + c$

ii) The Cauchy problem

$\begin{cases} u_x + e^x u_y = 1 \\ u(x,0) = 1 \quad \text{on } \Gamma: y = 0 \end{cases}$

Assume that the characteristic passes through $P(s,0,1)$ at time $t = 0$.

$\begin{cases} x = a = s \\ y = e^a + b = 0 \Rightarrow b = -e^s \\ u = c = 1 \end{cases}$

Thus, the characteristic are

$\begin{cases} x = t + s \longrightarrow s = x - t \\ y = e^{t+s} - e^s \rightarrow y = e^s(e^t - 1) \\ u = t + 1 \rightarrow t = u - 1 \end{cases}$

$\Rightarrow y = e^{x-t}(e^t - 1)$

and $y = e^{x-u+1}(e^{u-1} - 1) = e^x - e^{x-u+1}$

$e^{x-u+1} = e^x - y$

$\boxed{u(x,y) = x + 1 - \ln(e^x - y), \quad y < e^x}$

iii) Observe that

$y = e^x$ is a characteristic curve.

Let $u(x,y) = x$ and

$u(x,y) = x + y - e^x$.

Both functions are solutions to $u_x + e^x u_y = 1$ and they satisfy the Cauchy data.

iv) The general solution of the DE is

$u(x,y) = x + f(y - e^x)$, where

f is a differentiable function.

If a solution $u(x,y)$ satisfies the Cauchy data $u(x,e^x) = 1$, then, we should find f such that

$u(x, e^x) = x + f(e^x - e^x) = 1$

$x + f(0) = 1$

$f(0) = \frac{1}{x}$, for every x.

Impossible.

There is no solution satisfying $u(x, e^x) = 1$.

2.) $u_{xx} + \alpha u_{xy} + 7 u_{yy} = 0$

a) $B^2 - AC = \left(\frac{\alpha}{2}\right)^2 - 7 = \frac{\alpha^2 - 28}{4}$

for $\alpha \in (-\infty, -\sqrt{28}) \cup (\sqrt{28}, \infty)$: Hyperbolic

for $\alpha \in (-\sqrt{28}, \sqrt{28})$: Elliptic

for $\alpha = -\sqrt{28}, \sqrt{28}$: Parabolic

b) $u_{xx} - 8 u_{xy} + 7 u_{xy} = 0$

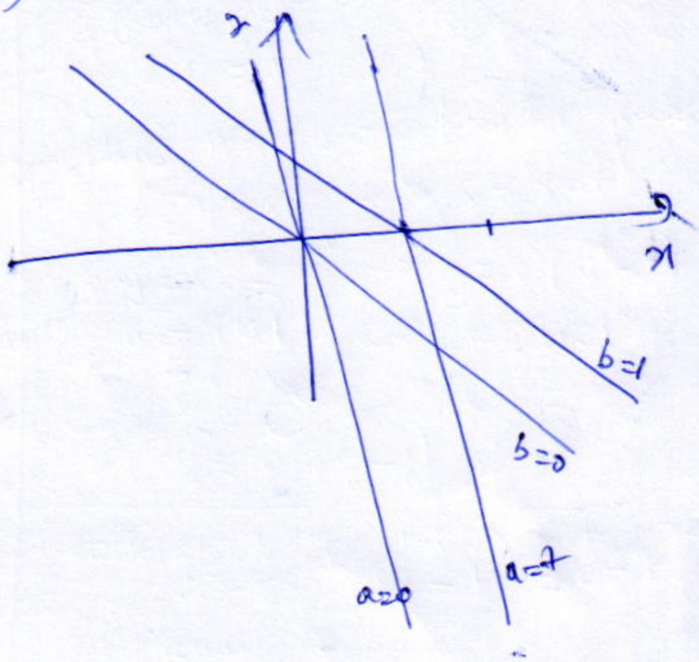
i) Characteristics are

$$\frac{dy}{dx} = \frac{-4 - \sqrt{\frac{36}{4}}}{1} = -7$$

and

$$\frac{dy}{dx} = \frac{-4 + \ }{1} = -1$$

ii) $y = -7x + a$ and $y = -x + b$



(iii) $\xi = y + 7x$, $\eta = y + x$.

$u_x = w_\xi \xi_x + w_\eta \eta_x$

$= 7 w_\xi + w_\eta$

$$u_{xx} = 7(w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x) + (w_{\xi\eta}\xi_x + w_{\eta\eta}\eta_x)$$
$$= 49 w_{\xi\xi} + 14 w_{\xi\eta} + w_{\eta\eta}$$
$$u_y = w_\xi \xi_y + w_\eta \eta_y$$
$$= w_\xi + w_\eta$$
$$u_{yy} = (w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y) + (w_{\xi\eta}\xi_y + w_{\eta\eta}\eta_y)$$
$$= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$
$$u_{xy} = 7(w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y) + (w_{\xi\eta}\xi_y + w_{\eta\eta}\eta_y)$$
$$= 7w_{\xi\xi} + 8w_{\xi\eta} + w_{\eta\eta}$$
$$\Rightarrow 49\cancel{w_{\xi\xi}} + 14w_{\xi\eta} + \cancel{w_{\eta\eta}} - 8(7\cancel{w_{\xi\xi}} + 8w_{\xi\eta} + \cancel{w_{\eta\eta}})$$
$$+7(\cancel{w_{\xi\xi}} + 2w_{\xi\eta} + \cancel{w_{\eta\eta}}) = 0$$
$$-36 w_{\xi\eta} = 0$$
$$\Rightarrow w_{\xi\eta} = 0 \Rightarrow w(\xi,\eta) = f(\xi) + g(\eta)$$
and, $\boxed{u(x,y) = f(y+7x) + g(x+y)}$,
where f, g are twice differentiable functions

Verification

$$u_x = 7f'(7x+y) + g'(x+y)$$
$$u_{xx} = 49f''(7x+y) + g''(x+y)$$
$$u_y = f'(y+7x) + g'(x+y)$$
$$u_{yy} = f''(7x+y) + g''(x+y)$$
$$u_{xy} = 7f''(7x+y) + g''(x+y)$$
$$\Rightarrow u_{xx} - 8u_{xy} + 7u_{yy} = 0$$

$$u_{xx} + \sqrt{28}\, u_{xy} + 7u_{yy} = 0$$

c.) Characteristic equation

$$\frac{dy}{dx} = \frac{\sqrt{28}}{2} = \sqrt{7}$$
$$y = \sqrt{7}x + C$$
$$\xi = y - \sqrt{7}x, \quad \eta = x, \quad J = \left| \quad \right| =$$
$$u_x = w_\xi \xi_x + w_\eta \eta_x$$
$$= -\sqrt{7}w_\xi + w_\eta$$
$$u_{xx} = -\sqrt{7}(w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x) + (w_{\xi\eta}\xi_x + w_{\eta\eta}\eta_x)$$
$$= 7w_{\xi\xi} - 2\sqrt{7}w_{\xi\eta} + w_{\eta\eta}$$
$$u_y = w_\xi \xi_y + w_\eta \eta_y$$
$$= w_\xi$$
$$u_{yy} = w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y = w_{\xi\xi}$$
$$u_{xy} = -\sqrt{7}(w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y) + (w_{\xi\eta}\xi_y + w_{\eta\eta}\eta_y)$$
$$= -\sqrt{7}w_{\xi\xi} + w_{\xi\eta}$$
$$\Rightarrow 7\cancel{w_{\xi\xi}} - \cancel{2\sqrt{7}w_{\xi\eta}} + w_{\eta\eta} + \sqrt{28}(\cancel{-\sqrt{7}w_{\xi\xi}} + \cancel{w_{\xi\eta}})$$
$$+7\cancel{w_{\xi\xi}} = 0$$
$$w_{\eta\eta} = 0$$
$$\Rightarrow w(\xi,\eta) = \eta f(\xi) + g(\xi)$$
$$\boxed{u(x,y) = x f(y-\sqrt{7}x) + g(y-\sqrt{7}x)},$$
f, g are twice differentiable functions

Verification

$$u_x = f(y-\sqrt{7}x) - \sqrt{7}x f'(y-\sqrt{7}x) - \sqrt{7}g'(y-\sqrt{7}x)$$
$$u_{xx} = -2\sqrt{7}f' + 7xf'' + 7g''$$
$$u_y = xf'(y-\sqrt{7}x) + g'$$
$$u_{yy} = xf'' + g''$$
$$u_{xy} = f' - \sqrt{7}xf'' - \sqrt{7}g''$$

$\Rightarrow u_{xx} + \sqrt{28}\, u_{xy} + 7u_{yy} = 0$

d) $u_{xx} + \sqrt{3}\, u_{xy} + 7u_{yy} = 0$

Elliptic DE.

There is no characteristic equation.

However, let $\eta = x + y$

$$\frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2} + 7}{1 + \frac{\sqrt{3}}{2}} = \frac{14 + \sqrt{3}}{2 + \sqrt{3}} = 25 - 12\sqrt{3}$$

$$y = (25 - 12\sqrt{3})x + C$$

$$\xi = y - (25 - 12\sqrt{3})x$$

$$J = \begin{vmatrix} -(25-12\sqrt{3}) & 1 \\ 1 & 1 \end{vmatrix} \neq 0$$

$$u_x = w_\xi \xi_x + w_\eta \eta_x$$
$$= -(25-12\sqrt{3}) w_\xi + w_\eta$$

$$u_{xx} = -(25-12\sqrt{3})\left[w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x\right] + \boxed{(w_{\xi\eta}\xi_x + w_{\eta\eta}\eta_x)}$$
$$= (25-12\sqrt{3})^2 w_{\xi\xi} - 2(25-12\sqrt{3}) w_{\xi\eta} + w_{\eta\eta}$$

$$u_y = w_\xi \xi_y + w_\eta \eta_y$$
$$= w_\xi + w_\eta$$

$$u_{yy} = w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y + w_{\xi\eta}\xi_y + w_{\eta\eta}\eta_y$$
$$= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$

$$u_{xy} = -(25-12\sqrt{3})\left[w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y\right] + \left[w_{\xi\eta}\xi_y + w_{\eta\eta}\eta_y\right]$$
$$= -(25-12\sqrt{3}) w_{\xi\xi} - (24-12\sqrt{3}) w_{\xi\eta} + w_{\eta\eta}$$

$$\Rightarrow (25-12\sqrt{3})^2 w_{\xi\xi} - 2(25-12\sqrt{3})\cancel{w_{\xi\eta}} + w_{\eta\eta}$$
$$+ \sqrt{3}\left[-(25-12\sqrt{3}) w_{\xi\xi} - (24-12\sqrt{3})\cancel{w_{\xi\eta}} + w_{\eta\eta}\right]$$
$$+ 7(w_{\xi\xi} + 2\cancel{w_{\xi\eta}} + w_{\eta\eta}) = 0$$

$$\left[7 - \sqrt{3}(25-12\sqrt{3}) + (25-12\sqrt{3})^2\right] w_{\xi\xi} + (8+\sqrt{3}) w_{\eta\eta} =$$

8.) $\begin{cases} u_{xx} - 4u_{xy} + y\, u_{yy} + u_x + y\, u_y = 0 \\ u(0,y) = y^3, \quad u_x(0,y) = 4y \end{cases}$

$u_y(0,y) = 3y^2, \quad u_{xy}(0,y) = 4$

$u_{yy}(0,y) = 6y,$

The Taylor series around the y-axis is

$$u(x,y) = u(0,y) + x\, u_x(0,y) + \frac{x^2}{2} u_{xx}(0,y) + \frac{1}{6} x^3 u_{xxx}(0,y) + \cdots$$

From (1), $u_{xx}(0,y) = 4(4) - y(6y) - 4y - 3y^3$
$$= 16 - 4y - 6y^2 - 3y^3.$$

We differentiate (1)

$$u_{xxx} - 4u_{xxy} + y\, u_{xyy} + u_{xx} + y\, u_{xy} = 0$$

But $u_{xyy} = 0$

$$u_{xxy} = -4 - 12y - 9y^2$$

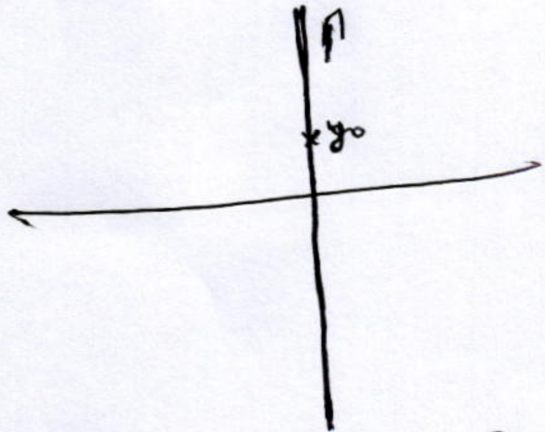
$$\Rightarrow u_{xxx}(0,y) = 4(-4 - 12y - 9y^2) - (16 - 4y - 6y^2 - 3y^3) - y(4)$$
$$= -32 - 48y - 30y^2 + 3y^3$$

$$\Rightarrow \boxed{u(x,y) = y^3 + 4xy + \frac{x^2}{2}(16 - 4y - 6y^2 - 3y^3) + \frac{x^3}{6}(-32 - 48y - 30y^2 + 3y^3) + \cdots}$$

3.| a.) The characteristic equations are

$$\frac{dy}{dx} = -2 \pm \sqrt{4-y}$$

The y-axis has equation $x = 0$.



Any point y_0 on Γ has a vertical tangent.

But, any point y_0 on a characteristic curve has a tangent of slope $\left.\frac{dy}{dx}\right|_{y=y_0} = -2 \pm \sqrt{4-y_0}$, which exclude vertical tangents.

Consequently, $x=0$ is not a characteristic curve for the equation.