King Fahd University of Petroleum & Minerals Name
Department of Mathematics & Statistics
MATH 455/ Final Exam/ 122 ID #

---

1. [20pts] (a) Prove that $x^2 + y^2 = 11(z^2 + w^2)$ has no nontrivial integer solution.
(b) Using the identity $(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$ determine all integer solutions of the equation $x^2 + y^2 = 2z^2$
(c) Show that if $x^3 + 2y^3 + 4z^3 \equiv 6xyz (\mathrm{mod}\, 7)$ then $x \equiv y \equiv z \equiv 0 (\mathrm{mod}\, 7)$. Deduce that the equation $x^3 + 2y^3 + 4z^3 - 6xyz = 0$ has no nontrivial integer solutions.

---

2. **[20pts]** (a) Let the polynomial equation with integer coefficients $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$, where $c_n \neq 0$, have a nonzero rational solution $\frac{a}{b}$ with a and b coprime integers. Prove that $a|c_0$ and $b|c_n$.
(b) Let $a \in \mathbb{N}$. Prove that $\sqrt{a+1} - \sqrt{a}$ is irrational.
(c) Find, in terms of the prime p, the integer(s) a for which $\sqrt{a+p} - \sqrt{a}$ is rational. What then are the possible values of the rational number $\sqrt{a+p} - \sqrt{a}$?

---

3. [20pts] (a) Let a, b, x, y be integers such that $(a-b) \,|\, (ax+y)$. Prove that $(a-b) \,|\, (bx+y)$.
(b) Show that if n is the product of odd distinct primes $p_1, \ldots, p_r$ $(r \geq 2)$ and each $p_i - 1$ divides $n - 1$, then n is a Carmichael number.
(c) Find all Carmichael numbers of the form $65p$ where p is prime.

---

4. [20pts] (a) Determine the number of solutions of the congruence $x^{20} \equiv 13 (\mathrm{mod}\, 17)$. What is the number of solutions of $x^{20} \equiv 13 (\mathrm{mod}\, 51)$?
(b) Let a and k be integers greater than 1. Determine the order of a modulo $(a^k - 1)$. Is it true that $k|\phi\left(a^k - 1\right)$? Justify.
(c) State Wilson's theorem and prove that if p is prime and k is an integer such that $1 \leq k < p$, then $(p-k)!\,(k-1)! \equiv (-1)^k \,(\mathrm{mod}\, p)$ and that $(2p-1)! \equiv p \,(\mathrm{mod}\, p^2)$

---

5. [20pts] (a) State and prove Möbius inversion formula.
(b) Let $f(n)$ be an arithmetic function. Show that $\sum_{d|n} f(d) = n$ for all $n \in \mathbb{N}$ if and only if $f(n) = \phi(n)$
(c) Show that for all $n \in \mathbb{N}$, $\sum_{d|n} \mu(d)\phi(d) = (-1)^{\omega(n)} \prod_{p|n} (p-2)$.