KFUPM - Department of Mathematics and Statistics
MATH 345, Term 122
Final Exam (Out of 100), Duration: 180 minutes

NAME:
ID:

Solve the following Exercises.

Exercise 1 (10 points 4-3-3): Let G be the multiplicative group of all invertible 2×2 matrices, H the subgroup of G consisting of all 2×2 matrices A such that $detA = 1$.
(1) Find the center $\mathcal{Z}(G)$ of G.
(2) Prove that $\mathcal{Z}(H) = H \cap \mathcal{Z}(G)$.
(3) True or false: Any subgroup of a non-abelian group is not abelian. Justify.

Exercise 2 (15 points 5-5-5): Let G be a finite abelian group with $|G| = pq$ where p and q are distinct positive prime integers. Set $H_1 = \{x \in G | x^q = e\}$ and $H_2 = \{x \in G | x^p = e\}$.
(1) Prove that H_1 and H_2 are incomparable under the inclusion.
(2) Are H_1 and H_2 cyclic groups? Find their orders.
(3) Prove that G is the internal direct product of H_1 and H_2.

Exercise 3 (15 points 5-5-5): Let R be a commutative ring with unity, I an ideal of R and set $\sqrt{I} = \{a \in R | a^n \in I \text{ for some positive integer } n\}$.
(1) Prove that $\sqrt{I}$ is an ideal of R containing I.
(2) Prove that if P is a prime ideal of R, then $P = \sqrt{P}$.
(3) If $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$, find $\sqrt{I}$.

Exercise 4 (15 points 5-5-5): Let R be a commutative ring and $\phi : R[X] \longrightarrow R$ defined by $\phi(f) = a_0$ whenever $f = a_0 + a_1 X + \cdots + a_n X^n$.
(1) Prove that ϕ is a ring homomorphism. Is ϕ onto?
(2) Find the kernel $Ker(\phi)$ and show that $R[X]/(X)$ is isomorphic to R ((X) is the ideal of $R[X]$ generated by X).
(3) Under which condition (X) is a prime ideal of $R[X]$ (resp. a maximal ideal of $R[X]$)?

Exercise 5 (10 points, 5-5): Let R be an integral domain.
Prove that $R[X]$ is a PID (Principal Ideal Domain) if and only if R is a field.

Exercise 6 (15 points, 5-5-5): Let $f = a + bX + X^2$ be a polynomial in $\mathbb{Z}[X]$.
(1) Prove that f is reducible over $\mathbb{Z}$ if and only if $f(x)$ has a root in $\mathbb{Z}$.
(2) Let p be a positive prime integer and set $g = a + bX + pX^2$ where p do not divide ab. Prove that g is reducible over $\mathbb{Z}$ only if g has a root in $\mathbb{Z}$.
(3) Application: Use (2) to prove that $2X^2 + X + 1$ is irreducible over $\mathbb{Z}$. Is the Mod p Test applicable? Justify.

Exercise 7 (10 points, 5-5): Let D be an integral domain.
(1) Prove that every prime element of D is irreducible.
(2) Assume that D is a PID. Prove that every irreducible element of D is prime.

Exercise 8 (10 points, 5-5): Let $D = \mathbb{Z}[\sqrt{-5}]$.
(1) Prove that $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are irreducible.
(2) Is D a UFD (Unique Factorization Domain)? Justify.