KFUPM - Department of Mathematics and Statistics
MATH 345, Term 122
Exam III (Out of 80), Duration: 120 minutes

NAME:
ID:

Solve the following Exercises.

Exercise 1 (15 points 5-5-5): Let R and S be subrings of a ring T.
(1) Prove that $R \cap S$ is a subring of T.
(2) Under which condition $R \cup S$ is a subring of T?
(3) Find an example of a ring T with two subrings R and S such that $R \cup S$ is not a subring of T.

Exercise 2 (15 points 5-5-5): Let R be a commutative ring satisfying the property $(\mathbf{P})$: For every $0 \neq a \in R$, and for every $b, c \in R$, $ab = ac \Rightarrow b = c$.
(1) Is R an integral domain? Justify.
(2) If R has a unity element, is R an integral domain?
(3) Give an example of a commutative ring R which satisfies the property $(\mathbf{P})$ but which is not an integral domain.

Exercise 3 (10 points 5-5): Let R be a commutative ring with unity.
(1) Prove that if R is of characteristic zero, then $\mathbb{Z}$ is isomorphic to a subring of R.
(2) Prove that if R is of prime characteristic p, then $\mathbb{Z}/p\mathbb{Z}$ is isomorphic to a subring of R.

Exercise 4 (10 points 5-5): Let R be a commutative ring and M and N two distinct maximal ideals of R.
(1) Prove that $M + N = R$, and $M \cap N = MN$.
(2) Give an example of a commutative ring with two maximal ideals M and N such that $M + N = R$.

Exercise 5 (15 points, 5-5-5):
Let R be an integral domain and K its field of fractions.
(1) Prove that K is the smallest field containing R.
(2) Prove that K is the intersection of all fields containing R.
(3) What is the field of fractions of the ring $R = \mathbb{Z}(\sqrt{2}) = \{a + b\sqrt{2}, a, b \in \mathbb{Z}\}$.

Exercise 6 (15 points, 5-5-5):
Let $R = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2}, a, b \in \mathbb{Q}\}$, and $S = \mathbb{Q}(i) = \{a + ib, a, b \in \mathbb{Q}\}$.
(1) Prove that R is a field.
(2) Prove that S is a field.
(3) Are R and S comparable.