KFUPM - Department of Mathematics and Statistics
MATH 345, Term 122
Exam II (Out of 80), Duration: 120 minutes

NAME:
ID:

Solve the following Exercises.

Exercise 1 (10 points 3-3-4):
(1) Find all elements of order 2 of the multiplicative group $\mathbb{C}^*$.
(2) Find all elements of order 2 of the group $\mathbb{R}^* \bigoplus \mathbb{R}^*$.
(3) Use (1) and (2) to show that $\mathbb{C}^*$ is not isomorphic to $\mathbb{R}^* \bigoplus \mathbb{R}^*$.

Exercise 2 (20 points 5-5-5-5):
(1) Find $Aut(\mathbb{Z})$, the group of all automorphisms of the additive group $(\mathbb{Z}, +)$.
(2) Let H be a (multiplicative) cyclic group of order 3. Find $Aut(H)$.
(3) Prove that $Aut(H)$ is isomorphic to $Aut(\mathbb{Z})$.
(4) Let G and G' be two groups such that $Aut(G)$ is isomorphic to $Aut(G')$. Are G and G' isomorphic? Justify.

Exercise 3 (20 points 7-8-5): (1) Find $Aut(\mathbb{Q})$, the group of all automorphisms of the additive group $(\mathbb{Q}, +)$.
(2) Prove that $Aut(\mathbb{Q})$ is isomorphic to the multiplicative group $(\mathbb{Q}^*, \times)$.
(3) Prove that the additive group $\mathbb{Q}$ has no proper subgroup of finite index.

Exercise 4 (15 points 5-5-5): Let G be the external direct product of the groups $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/5\mathbb{Z}$, that is, $G = \mathbb{Z}/3\mathbb{Z} \bigoplus \mathbb{Z}/4\mathbb{Z} \bigoplus \mathbb{Z}/5\mathbb{Z}$.
(1) Is G a cyclic group?
(2) Is G isomorphic to $\mathbb{Z}/4\mathbb{Z} \bigoplus \mathbb{Z}/15\mathbb{Z}$?
(3) Is $\mathbb{Z}/3\mathbb{Z} \bigoplus \mathbb{Z}/3\mathbb{Z} \bigoplus \mathbb{Z}/4\mathbb{Z} \bigoplus \mathbb{Z}/5\mathbb{Z}$ isomorphic to $\mathbb{Z}/3\mathbb{Z} \bigoplus \mathbb{Z}/4\mathbb{Z} \bigoplus \mathbb{Z}/15\mathbb{Z}$?

Exercise 5 (15 points, 4-7-4):
Let $G = \mathcal{M}_n(\mathbb{R})$ be the additive group of all $n \times n$ matrices, H_1 the set of all $n \times n$ symmetric matrices (i. e. $A = A^T$) and H_2 be the set of all $n \times n$ skew symmetric matrices (i. e. $A = -A^T$).
(1) Prove that H_1 and H_2 are subgroups of G.
(2) Prove that G is the internal direct Product of H_1 and H_2.
(3) Are H_1 and H_2 isomorphic?