King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
(Math 260)

First Exam
Term 122
Monday, March 04, 2013
Net Time Allowed: 120 minutes

Name:	
ID:	
Section No:	

(Show all your steps and work)

Q		Points
1	Key_	10
2		10
3		10
4		12
5	Solutions	10
6		10
7		10
8		7
9		7
10		7
11		7
Total		**100**

(1) [10 points]

(a) Verify that $y(x) = c_1 x^{-3} + c_2 x^{-1}$, where c_1 and c_2 are arbitrary constants, is a solution of $x^2 y'' + 5xy' + 3y = 0$

$$y'(x) = -3c_1 x^{-4} - c_2 x^{-2}$$ 2

$$y''(x) = 12C_1 x^{-5} + 2C_2 x^{-3}$$ 2

$$x^2 y'' + 5x y' + 3y = x^2(12C_1 x^{-5} + 2C_2 x^{-3}) + 5x(-3C_1 x^{-4} - C_2 x^{-2}) + 3(C_1 x^{-3} + C_2 x^{-1})$$ 2

$$= x^{-3}(12C_1 - 15C_1 + 3C_1) + x^{-1}(2C_2 - 5C_2 + 3C_2)$$

$$= 0$$

(b) Find the solution that satisfies $y(1) = 0$ and $y'(1) = 1$

$$y(x) = C_1 x^{-3} + C_2 x^{-1}$$

$$y'(x) = -3C_1 x^{-4} - C_2 x^{-2}$$

$$y(1) = C_1 + C_2 = 0 \quad -(1)$$ 1

$$y'(1) = -3C_1 - C_2 = 1 \quad -(2)$$ 1

Add (1) and (2) $\quad -2C_1 = 1 \Rightarrow C_1 = -\frac{1}{2}$

$$C_2 = \frac{1}{2}$$ 2

$$\boxed{y(x) = -\frac{1}{2} x^{-3} + \frac{1}{2} x^{-1}}$$

2 (3) Solve the differential equation $\left(1-x^2\right)\frac{dy}{dx}=2y$ [10 points]

separating variables

$$\frac{dy}{y} = \frac{2}{1-x^2}\,dx$$ 2

Using partial fractions $\frac{2}{1-x^2} \equiv \frac{1}{1+x} + \frac{1}{1-x}$ 2

Hence $$\frac{dy}{y} = \left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$$

Integrating both sides

$$\int \frac{dy}{y} = \int\left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$$

$$\ln|y| = \ln|(1+x)| - \ln|(1-x)| + \ln C$$ 2

$$= \ln\left|C\,\frac{1+x}{1-x}\right|$$ 2

Hence $$\boxed{y = C\left(\frac{1+x}{1-x}\right)}$$

2

Constant worth 1-point

In any Question, if the students miss the constant C, award him (-1) !

3 Solve the Initial Value Problem $y' - \frac{y}{1+2x} = 1,\ y(0) = 2$ [10 points]

This is linear D.E with $P(x) = -\frac{1}{1+2x}$ 2

$$\text{I.F.} = e^{\int P(x)dx} = e^{-\int \frac{1}{1+2x}dx} = e^{-\frac{1}{2}\int \frac{2}{1+2x}dx}$$

$$= e^{-\frac{1}{2}\ln|1+2x|} = (1+2x)^{-1/2}$$ 2

Multiply throughout by $(1+2x)^{-1/2}$

$$(1+2x)^{-1/2} y' - \frac{y}{(1+2x)^{3/2}} = (1+2x)^{-1/2}$$

or $$\frac{d}{dx}\left[y\,(1+2x)^{-1/2} \right] = (1+2x)^{-1/2}$$ 2

Integration gives

$$y(1+2x)^{-1/2} = \cancel{2} \cdot \frac{1}{\cancel{2}} (1+2x)^{1/2} + C$$

2 so that $$y = (1+2x) + C(1+2x)^{1/2}$$

Now $y(0) = 2$, so

$$2 = 1 + C \cdot 1 \Rightarrow C = 1.$$

Hence the solution is

$$\boxed{y = (1+2x) + (1+2x)^{1/2}}$$ 2

4 Solve the differential Equation $\dfrac{dy}{dx} = \dfrac{4x+y}{x-4y}$ [~~10~~ 12 points]

This is homogeneous D.E.

Put $y = ux$ so that $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

The DE then gives $u + x\dfrac{du}{dx} = \dfrac{4x+ux}{x-4ux}$

$$= \frac{u+4}{1-4u}$$

or $x\dfrac{du}{dx} = \dfrac{u+4}{1-4u} - u$

$$= \frac{\cancel{u}+4-\cancel{u}+4u^2}{1-4u} = 4\,\frac{u^2+1}{1-4u}$$

Separating variables

$$\frac{1-4u}{u^2+1}\,du = \frac{4}{x}\,dx$$

or $\left(\dfrac{1}{u^2+1} - 4\dfrac{u}{u^2+1}\right)du = \dfrac{4}{x}\,dx$

Integrate both sides

$$\int\left(\frac{1}{u^2+1} - 2\,\frac{2u}{u^2+1}\right)du = \int\frac{4}{x}\,dx$$

or $\tan^{-1}u - 2\ln(u^2+1) = 4\ln x + \ln c$

$$= \ln cx^4$$

Rearranging

$$\tan^{-1}u = \ln cx^4(u^2+1)^2$$

As $u = \dfrac{y}{x}$,

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln cx^4\left(\frac{y^2}{x^2}+1\right)^2$$

or $\boxed{\tan^{-1}\left(\dfrac{y}{x}\right) = \ln c\,(x^2+y^2)^2}$

5 Solve $\left[\sin(xy)+xy\cos(xy)+2x\right]dx+\left[x^2\cos(xy)+2y\right]dy=0$

[10 points]

Here $M = \sin(xy) + xy\cos(xy) + 2x$ 1

$N = x^2\cos(xy) + 2y$ 1

Now $\dfrac{\partial M}{\partial y} = \underbrace{x\cos(xy) + x\cos(xy)} - x^2 y\sin(xy)$

$\dfrac{\partial N}{\partial x} = 2x\cos(xy) - x^2 y\sin(xy)$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, The D.E is Exact. 2

To find $\phi(x,y)$:

2 $\dfrac{\partial \phi}{\partial x} = M = \sin(xy) + xy\cos(xy) + 2x$ —(1)

$\dfrac{\partial \phi}{\partial y} = N = x^2\cos(xy) + 2y$ ——(2)

Integrate (2) w.r.t. y (partially)

$\phi(x,y) = x\sin(xy) + y^2 + f(x)$

Take partial derivative w.r.t x

2 $\dfrac{\partial \phi}{\partial x} = \sin(xy) + xy\cos(xy) + f'(x)$ —(3)

Compare (1) and (3)

$\cancel{\sin(xy)} + \cancel{xy\cos(xy)} + 2x = \cancel{\sin(xy)} + \cancel{xy\cos(xy)} + f'(x)$

2 So, that $f'(x) = 2x \Rightarrow f(x) = x^2$

Hence $\phi(x,y) = x\sin(xy) + y^2 + x^2$

The solution is

$$\boxed{x\sin(xy) + x^2 + y^2 = C}$$

6 [redacted] [10 points]

a) Find the Reduced Echelon form of the Matrix $\begin{bmatrix} 1 & 3 & 15 & 7 \\ 2 & 4 & 22 & 8 \\ 2 & 7 & 34 & 17 \end{bmatrix}$

$$\xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & -2 & -8 & -6 \\ 0 & 1 & 4 & 3 \end{bmatrix}$$

$$-\tfrac{1}{2}R_2 \rightarrow \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 4 & 3 \end{bmatrix}; \quad \xrightarrow[R_3 - R_2]{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the desired Reduced echelon form.

(It depends on the steps: 6 pts)

b) Use part a) to solve the Linear System

$$x_1 + 3x_2 + 15x_3 = 7$$
$$2x_1 + 4x_2 + 22x_3 = 8$$
$$2x_1 + 7x_2 + 34x_3 = 17$$

From above echelon form, x_3 is free. Put $x_3 = t$

Row 1 $\Rightarrow$ $x_1 + 3x_3 = -2 \Rightarrow x_1 = -2 - 3t$

Row 2 $\Rightarrow$ $x_2 + 4x_3 = 3 \Rightarrow x_2 = 3 - 4t$

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 - 3t \\ 3 - 4t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} t$ 4-pts bold

all values of k

inconsistent

7 Find all values of k such that the following system is an Inconsistent system. [10 points]

$$x+2y+z=3$$
$$2x-y-3z=5$$
$$4x+3y-z=k$$

Consider the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & -1 & -3 & 5 \\ 4 & 3 & -1 & K \end{bmatrix} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -5 & -5 & -1 \\ 0 & -5 & -5 & K-12 \end{bmatrix}$$

$$-\frac{1}{5}R_2 \rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & \frac{1}{5} \\ 0 & -5 & -5 & K-12 \end{bmatrix}$$

$$\xrightarrow[R_3+5R_2]{R_1-2R_2} \begin{bmatrix} 1 & 0 & -1 & \frac{13}{5} \\ 0 & 1 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & K-11 \end{bmatrix}$$

The system will be inconsistent for all $K \neq 11$.

* Row-operations: 6-pts

Last step where $k-11 \neq 0$, 4-pts

(8) If the Gaussian-Jordan method is used to solve the linear system: [7 points]

$$\begin{cases} y = -2x - 2z + 1 \\ x = -2y - z + 2 \\ z = x - y, \end{cases}$$

We get the matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$, then $a+b+c=$

A) 1

B) 2

C) -3

D) -2

E) 3

7

(9) Which one of the following statements is "False" about the nxn matrices A, B and C:

[7 points]

A) $(AB)C = A(BC)$

B) $(A+B)C = AC + BC$

C) $(A+B)^2 = A^2 + AB + BA + B^2$

D) If $AB = 0$, then either $A = 0$ *or* $B = 0$

E) $(A-B)^2 = A^2 - AB - BA + B^2$

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(10) If

$$A=\begin{bmatrix}2\\4\end{bmatrix},\ B=\begin{bmatrix}1/2 & 1/2 & 1\end{bmatrix},\ C=\begin{bmatrix}0 & 1\\0 & 2\\2 & 3\end{bmatrix},\ D=\begin{bmatrix}5/9 & -1\\-8/9 & -1\end{bmatrix}, \textit{then } (AB)\left(\frac{1}{9}C\right)+D=$$

[7 points]

A) $\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}$

B) $\begin{bmatrix}4/9 & 1\\4/9 & 2\end{bmatrix}$

C) $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

D) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

E) $\begin{bmatrix}4/9 & 1\\0 & 1\end{bmatrix}$

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(11) If $B=\begin{bmatrix}a & 1\\a & 0\end{bmatrix}$ and $B^2+B+I=0$ then $a=$ [7 points]

A) -2

B) 0

C) 1

D) 2

E) -1

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