

**King Fahd University of Petroleum and Minerals**  
**Department of Mathematics and Statistics**

**FINAL EXAM – MATH 202 (Term 122)**

May 27, 2013

Duration: **180** Minutes

Name : SOLUTION KEY

ID# : \_\_\_\_\_

Section # : \_\_\_\_\_

Serial #: \_\_\_\_\_

**Please read the following:**

1. Exam has 2 parts: **Part I:** 10 MCQs, **Part II:** 5 Written Questions.
2. Provide **all necessary steps** with **clear writing** for **Part II**.
3. For **Part I (MCQ)**, credit will be given only for the correct answer posted **BELOW**.
4. **Mobiles** and **calculators** are **NOT allowed** in this exam.

**Part I: Multiple Choice [7 Points for each correct answer]**

MCQ #	Student Answer	Points
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
<b>Total Points /70</b>		

**Part II: Written**

Q #	Points	Max
11		14
12		14
13		14
14		16
15		12
<b>Total</b>		<b>70</b>

<b>Grand Total /140</b>	
-------------------------	--

**Part I: Multiple Choice**

MCQ #	CODE 001	CODE 002	CODE 003	CODE 004
1	D	E	A	C
2	C	D	E	C
3	C	B	A	A
4	B	C	B	E
5	D	D	D	B
6	D	E	B	A
7	B	C	A	B
8	A	B	D	C
9	C	D	B	D
10	E	A	C	A

**Part II: Written**

**Q11. (14 points)** Find two power series solutions of the differential equation  $y'' + x^2y' + xy = 0$  about the ordinary point  $x = 0$ . Give the first three nonzero terms for each series solution.

**Hint:** The assumption  $y = \sum_{n=0}^{\infty} c_n x^n$  and its first two derivatives lead to

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=2}^{\infty} (k-1) c_{k-1} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k = 0. \quad (1)$$

**Solution:** Equation (1) can be written as

$$2c_2 + 6c_3x + c_0x + \sum_{k=2}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=2}^{\infty} (k-1) c_{k-1} x^k + \sum_{k=2}^{\infty} c_{k-1} x^k = 0$$

$$\text{or } 2c_2 + (c_0 + 6c_3)x + \sum_{k=2}^{\infty} [(k+2)(k+1) c_{k+2} + k c_{k-1}] x^k = 0.$$

We have  $c_2 = 0$ ,  $c_3 = -\frac{c_0}{6}$  and

$$c_{k+2} = -\frac{k c_{k-1}}{(k+2)(k+1)}, \quad k = 2, 3, 4, \dots \quad (2)$$

This relation generates consecutive coefficients of the assumed solution as we let  $k$  take on the successive integers indicated in (2):

$$k = 2, \quad c_4 = -\frac{2c_1}{4 \cdot 3} = -\frac{c_1}{6}$$

$$k = 3, \quad c_5 = -\frac{3c_2}{5 \cdot 4} = 0$$

$$k = 4, \quad c_6 = -\frac{4c_3}{6 \cdot 5} = \frac{4c_0}{6^2 \cdot 5}$$

$$k = 5, \quad c_7 = -\frac{5c_4}{7 \cdot 6} = \frac{5c_1}{6^2 \cdot 7}$$

and so on. Now we substitute the coefficients just obtained into the original assumption

$$\begin{aligned} y &= c_0 + c_1x - \frac{c_0}{6}x^3 - \frac{c_1}{6}x^4 + \frac{4c_0}{6^2 \cdot 5}x^6 + \frac{5c_1}{6^2 \cdot 7}x^7 - \dots \\ &= c_0y_1(x) + c_1y_2(x), \end{aligned}$$

where

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{4}{6^2 \cdot 5}x^6 - \dots$$

$$y_2(x) = x - \frac{1}{6}x^4 + \frac{5}{6^2 \cdot 7}x^7 - \dots$$

**Q12. (14 points)** Determine singular points of the differential equation

$$x^2(x^2 - 1)^2y'' + 2x(x - 1)y' + y = 0.$$

Classify each singular point as regular or irregular.

**Solution:**

It should be clear that  $x = 0$ ,  $x = 1$ ,  $x = -1$  are singular points of the equation.

The standard form of the equation is

$$y'' + P(x)y' + Q(x)y = 0,$$

where

$$P(x) = \frac{2}{x(x-1)(x+1)^2} \quad \text{and} \quad Q(x) = \frac{1}{x^2(x+1)^2(x-1)^2}.$$

Since both rational functions

$$p(x) = xP(x) = \frac{2}{(x-1)(x+1)^2} \quad \text{and} \quad q(x) = x^2Q(x) = \frac{1}{(x+1)^2(x-1)^2}$$

are analytic at  $x = 0$ . We conclude that  $x = 0$  is a *regular singular point*.

Similarly, we are led to the conclusion that  $x = 1$  is a *regular singular point*. This follows from the fact that

$$p(x) = (x-1)P(x) = \frac{2}{x(x+1)^2} \quad \text{and} \quad q(x) = (x-1)^2Q(x) = \frac{1}{x^2(x+1)^2}$$

are both analytic at  $x = 1$ .

For  $x = -1$ ,

$$p(x) = (x+1)P(x) = \frac{2}{x(x-1)(x+1)}$$

is not analytic at  $x = -1$ . Thus,  $x = -1$  is an *irregular singular point*.

**Q13. (14 points)** Solve the initial value problem

$$\begin{aligned}\frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= -2x - y, \quad x(0) = 1, \quad y(0) = 1.\end{aligned}$$

**Solution:**

The characteristic equation of the system is

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ -2 & -1 - \lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

We find the eigenvalues  $\lambda_1 = i$  and  $\lambda_2 = -i$ .

For  $\lambda_1 = i$  the system

$$\begin{pmatrix} 1 - i & 1 \\ -2 & -1 - i \end{pmatrix} \begin{pmatrix} 1 - \lambda \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives  $k_2 = -(1 - i)k_1$ . By choosing  $k_1 = 1$ , we get

$$K_1 = \begin{pmatrix} 1 \\ -1 + i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We form the vectors

$$B_1 = \operatorname{Re}(K_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad B_2 = \operatorname{Im}(K_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] + c_2 \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t \right].$$

Imposing the initial conditions  $x(0) = 1$ ,  $y(0) = 1$  gives  $c_1 = 1$  and  $c_2 = 2$ .

Thus, the required solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t + 2 \sin t \\ \cos t - 3 \sin t \end{pmatrix}$$

**Q14. (16 points)** Find the general solution of the system

$$X' = \begin{pmatrix} 3 & 4 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & 3 \end{pmatrix} X.$$

**Solution:** From the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 4 & 0 \\ -1 & -\lambda & 2 \\ 0 & 2 & 3 - \lambda \end{vmatrix} = \lambda(3 - \lambda)^2 = 0,$$

we see that the eigenvalues are  $\lambda_1 = 0$  and  $\lambda_2 = 3$  (of multiplicity two). For  $\lambda_1 = 0$  Gauss-Jordan elimination gives

$$(A + 0I|\mathbf{0}) = \left( \begin{array}{ccc|c} 3 & 4 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 \end{array} \right) \xrightarrow{\text{row operations}} \left( \begin{array}{ccc|c} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Therefore  $k_1 = \frac{4}{3}k_2$ ,  $k_3 = -\frac{2}{3}k_2$ . The choice  $k_2 = -3$  gives an eigenvector

$$K_1 = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}.$$

Similarly, for  $\lambda_2 = 3$

$$(A + 3I|\mathbf{0}) = \left( \begin{array}{ccc|c} 0 & 4 & 0 & 0 \\ -1 & -3 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array} \right) \xrightarrow{\text{row operations}} \left( \begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

implies that  $k_1 = 2k_3$ ,  $k_2 = 0$ . Choosing  $k_3 = 1$  gives an eigenvector

$$K_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

We next solve the system  $(A + 3I)P = K_2$ :

$$\left( \begin{array}{ccc|c} 0 & 4 & 0 & 2 \\ -1 & -3 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{array} \right) \xrightarrow{\text{row operations}} \left( \begin{array}{ccc|c} 1 & 0 & -2 & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right).$$

We find  $p_1 = 2p_3 - \frac{3}{2}$ ,  $p_2 = \frac{1}{2}$ . Choosing  $p_3 = 0$  gives

$$P = \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

The general solution of the given equation is

$$X(t) = c_1 \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_3 \left[ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} t e^{3t} + \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} e^{3t} \right]$$

**Q15. (12 points)** Consider the nonhomogeneous system

$$X' = AX + \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

If the general solution of the associated homogeneous system  $X' = AX$  is

$$X_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t,$$

find a particular solution  $X_p$  of the nonhomogeneous system.

**Solution:**

The fundamental matrix of the system is given by

$$\Phi(t) = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix}$$

and

$$\Phi^{-1}(t) = \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}.$$

Here  $F(t) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ .

The required particular solution is

$$\begin{aligned} X_p &= \Phi(t) \int \Phi^{-1}(t)F(t)dt \\ &= \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \int \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} dt \\ &= \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \int \begin{pmatrix} -12 \\ 5e^{-t} \end{pmatrix} dt \\ &= \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \begin{pmatrix} -12t \\ -5e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} -12t - 15 \\ -12t - 10 \end{pmatrix} \end{aligned}$$