

King Fahd University of Petroleum and Minerals  
Department of Mathematics and Statistics

EXAM I – MATH 202 (Term 122)

February 26, 2013

Duration: 120 Minutes

Name: Solution key ID#: \_\_\_\_\_

Section/Instructor: \_\_\_\_\_ Serial #: \_\_\_\_\_

- Provide all necessary steps with clear writing.
- Mobiles and calculators are NOT allowed in this exam.

Question #	Marks	Maximum Marks
Q1		13
Q2		7
Q3		10
Q4		16
Q5		15
Q6		10
Q7		17
Q8		12
Total		100

Q1. Consider the differential equation  $(y^2 + y) - \frac{dy}{dx} = 0$ .

- (a) (5 points) Verify that  $y = \frac{Ce^x}{1 - Ce^x}$  is a one-parameter family of solutions of the given differential equation.

Solution:  $y = \frac{Ce^x}{1 - Ce^x} \Rightarrow \frac{dy}{dx} = \frac{Ce^x}{(1 - Ce^x)^2}$

$$\Rightarrow y^2 + y = \frac{C^2 e^{2x}}{(1 - Ce^x)^2} + \frac{Ce^x}{1 - Ce^x}$$

$$= \frac{C^2 e^{2x} + Ce^x(1 - Ce^x)}{(1 - Ce^x)^2} = \frac{-Ce^x}{(1 - Ce^x)^2}$$

Hence,  $y^2 + y - \frac{dy}{dx} = \frac{Ce^x}{(1 - Ce^x)^2} - \frac{Ce^x}{(1 - Ce^x)^2} = 0.$

Therefore,  $y = \frac{Ce^x}{1 - Ce^x}$  is a one-parameter family of solutions of the given DE.

- (b) (4 points) Find two constant solutions of the given differential equation.

Solution: Constant solutions can be obtained by solving the equation  $y^2 + y = 0.$

We have  $y = 0$  or  $y = -1.$

- (c) (4 points) Find a singular solution of the given differential equation.

Solution:

$y = 0$  can be determined from the family in (a) by choosing  $C = 0$

$y = -1$  cannot be obtained from the family.

Thus,

$y = -1$  is a singular solution of the given DE.

Q2. (7 points) Find the values of  $b$  such that the initial value problem

$$\frac{dy}{dx} = \sqrt{y-3x}, \quad y(2) = b$$

has a unique solution.

Solution :

$$\text{Here } f(x,y) = \sqrt{y-3x} \text{ and } f_y = \frac{1}{2\sqrt{y-3x}}.$$

Hence  $f$  and  $f_y$  are both continuous  
when  $y-3x > 0 \Rightarrow y > 3x$ . ]

For  $x_0 = 2$ , we must have  $y_0 > 3(2) = 6$ .

Therefore  $b > 6$ .

Q3. (10 points) Solve the separable differential equation

$$(3x + y + xy + 3)dx + (x^2 + 2x)dy = 0.$$

Solution:

The given equation can be written as

$$\left[ \begin{aligned} & [3(x+1) + y(x+1)] dx + (x^2 + 2x) dy = 0 \\ \Rightarrow & (3+y)(x+1) dx + (x^2 + 2x) dy = 0 \end{aligned} \right]$$

$$\Rightarrow \frac{x+1}{x^2+2x} dx + \frac{dy}{3+y} = 0$$

$$\Rightarrow \left( \frac{\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x+2} \right) dx + \frac{dy}{3+y} = 0$$

$$\Rightarrow \frac{1}{2} \ln|x| + \frac{1}{2} \ln|x+2| + \ln|3+y| = C_1$$

$$\Rightarrow (3+y) \sqrt{x(x+2)} = C$$

Q4. Consider the linear differential equation

$$xy' + (1+x)y = e^{-x} \cos^2 x.$$

- (a) (14 points) Find a solution of the equation that passes through the point  $(-\pi, 0)$ .

Solution: The standard form of the equation is

$$y' + \left(\frac{1+x}{x}\right)y = \frac{\cos^2 x}{xe^x}.$$

Here  $p(x) = \frac{1+x}{x}$ . The integrating factor is

$$e^{\int p(x) dx} = e^{\int \frac{1+x}{x} dx} = e^{\int \left(1 + \frac{1}{x}\right) dx} = e^{x + \ln x} = xe^x.$$

We obtain

$$\begin{aligned} \mathcal{L}(xe^x y) &= \cos^2 x \\ &= \frac{1}{2}(1 + \cos 2x) \end{aligned}$$

$$\begin{aligned} \Rightarrow xe^x y &= \frac{1}{2}\left(x + \frac{1}{2} \sin 2x\right) + C \\ \Rightarrow y &= \frac{1}{2e^x} + \frac{\sin 2x}{4xe^x} + \frac{C}{xe^x} \end{aligned} \quad ]$$

$$y(-\pi) = 0$$

$$\begin{aligned} \Rightarrow \left(\frac{1}{2} + \frac{C}{-\pi}\right)e^{\pi} &= 0 \\ \Rightarrow \frac{C}{\pi} = \frac{1}{2} \Rightarrow C &= \frac{\pi}{2} \end{aligned} \quad ]$$

So, the solution is

$$y = \frac{1}{2e^x} + \frac{\sin 2x}{4xe^x} + \frac{\pi}{2xe^x}$$

- (b) (2 points) Give the largest interval in which the solution in (a) is defined.

Solution:

The interval is

$$I = (-\infty, 0)$$



Q5. (15 points) Solve the initial value problem

$$ye^{2xy} + x + xe^{2xy} \frac{dy}{dx} = 0, \quad y(1) = 0$$

Solution :

The given equation can be written as

$$(ye^{2xy} + x) dx + xe^{2xy} dy = 0$$

Put  $M = ye^{2xy} + x$

$N = xe^{2xy}$

We have

$$M_y = e^{2xy} + 2xye^{2xy}$$

$$N_x = e^{2xy} + 2xye^{2xy}$$

Since  $M_y = N_x$ , the equation is an exact equation.

$$\begin{aligned} \Rightarrow F(x, y) &= \int M dx + g(y) \\ &= \int (ye^{2xy} + x) dx + g(y) \\ &= \frac{1}{2} e^{2xy} + \frac{1}{2} x^2 + g(y) \end{aligned}$$

$$\Rightarrow F_y = N \quad \Rightarrow \quad xe^{2xy} + g'(y) = xe^{2xy}$$

$$\Rightarrow g'(y) = 0$$

Put  $g(y) = 0.$

The general solution is  $\frac{1}{2} e^{2xy} + \frac{1}{2} x^2 = c_1$

$$\Rightarrow e^{2xy} + x^2 = c$$

Imposing the initial condition  $y(1) = 0$  gives  $c = 2.$

thus, The solution of the ivp is

$$e^{2xy} + x^2 = 2$$

- Q6. (10 points) Find an integrating factor that makes the differential equation  $xy^3 + y + (2x^2y^2 + 2x + 2y^4)y' = 0$  exact.  
(Note: **Do not solve** the new equation)

Solution:

The equation can be written as

$$(xy^3 + y)dx + (2x^2y^2 + 2x + 2y^4)dy = 0$$

Put  $M = xy^3 + y$

$$N = 2x^2y^2 + 2x + 2y^4.$$

We have

$$M_y = 3xy^2 + 1$$

$$N_x = 4xy^2 + 2$$

$$\begin{aligned} \Rightarrow \frac{N_x - M_y}{M} &= \frac{(4xy^2 + 2) - (3xy^2 + 1)}{xy^3 + y} \\ &= \frac{xy^2 + 1}{xy^3 + y} = \frac{1}{y} \end{aligned}$$

The integrating factor is

$$\begin{aligned} \mu(y) &= e^{\int \frac{N_x - M_y}{M} dy} \\ &= e^{\int \frac{1}{y} dy} \\ &= e^{\ln y} = y \end{aligned}$$

- Q7. (a) (10 points) Use an appropriate substitution to reduce the differential equation

$$\frac{dy}{dx} = y(xy^3 - 1)$$

to a linear equation. (Note: **Do not solve** the new equation)

Solution: The equation can be written as

$$\frac{dy}{dx} + y = xy^4.$$

It is a Bernoulli equation with  $n=4$ .

$$\text{Put } u = y^{1-n} = y^{-3} \Leftrightarrow y = u^{-1/3}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -\frac{1}{3} u^{-4/3} \frac{du}{dx}$$

Substituting  $u = y^{-3}$  gives

$$-\frac{1}{3} u^{-4/3} \frac{du}{dx} + u^{-1/3} = x u^{-4/3}$$

$$\Rightarrow \frac{du}{dx} - 3u = -3x,$$

- (b) (7 points) Use a suitable substitution to reduce the differential equation

$$\frac{dy}{dx} = \frac{x+y}{x+y+1}$$

to a separable equation. (Note: **Do not solve** the new equation)

$$\text{Solution: Put } u = x+y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$$

We obtain

$$\frac{du}{dx} - 1 = \frac{u}{u+1}$$

$$\Rightarrow \frac{du}{dx} = \frac{u}{u+1} + 1 = \frac{2u+1}{u+1}$$

$$\Rightarrow \frac{u+1}{2u+1} du = dx$$



- Q8. (12 points) A thermometer reading  $70^\circ F$  is taken from inside a room to the outside, where the air temperature is  $20^\circ F$ . Four minutes later, the thermometer reads  $30^\circ F$ . Find the thermometer's reading at  $t = 8$  minutes.

Solution: Newton's law of cooling/warming gives

$$\frac{dT}{dt} = k(T - T_m).$$

In this case,  $T_m = 20$ .

We have  $\frac{dT}{dt} = k(T - 20)$  or  $\frac{dT}{T - 20} = k dt$

$$\Rightarrow T(t) = 20 + Ce^{kt}.$$

We know that  $T(0) = 70$ ,

$$\Rightarrow 70 = 20 + C \Rightarrow C = 50,$$

$$\begin{aligned} \text{and } T(4) = 30 &\Rightarrow 30 = 20 + 50e^{4k} \\ &\Rightarrow e^{4k} = \frac{1}{5} \\ &\Rightarrow k = -\frac{\ln 5}{4}. \end{aligned}$$

Thus,

$$T(t) = 20 + 50e^{-\frac{\ln 5}{4}t}$$

and

$$\begin{aligned} T(8) &= 20 + 50e^{-2\ln 5} \\ &= 20 + 50\left(\frac{1}{25}\right) \\ &= 22. \end{aligned}$$

The thermometer's reading after 8 minutes is  $22^\circ F$ .