King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

EXAM II – MATH 202 (Term 122)

April 06, 2013

Duration: **120** Minutes

Name: ***Solution Key*** **ID#:** ________________

Section #: ________________________ **Serial #:** _____________

- Provide **all necessary steps** with **clear writing**.
- **Mobiles** and **calculators** are **NOT allowed** in this exam.

Question #	Marks	Maximum Marks
Q1		6
Q2		13
Q3		13
Q4		12
Q5		12
Q6		15
Q7		16
Q8		13
Total		**100**

Q1. (6 points) Find an interval centered about $x = 1$ for which the following initial value problem has a unique solution:

$$\left(x^2 - 4\right) y'' + 2xy' - 3y = 0,\ \ y(1) = 0,\ \ y'(1) = -1.$$

Solution:

We know that the functions $a_2(x) = x^2 - 4,\ a_1(x) = 2x$ and $a_0(x) = -3$ are all continuous on any interval containing $x = 1$,

and $a_2(x) = 0$ if $x = \pm 2$.

Thus, the required interval is $I = (0, 2)$.

Q2. (a) **(7 points)** The functions $y_1 = \sin(x^2)$ and $y_2 = \cos(x^2)$ are both solutions of the differential equation

$$xy'' - y' + 4x^3 y = 0.$$

Verify that y_1 and y_2 form a fundamental set of solutions of the given equation on the interval $(0, \infty)$.

Solution: The Wronskian of y_1, y_2 is

$$W(y_1, y_2) = \begin{vmatrix} \sin(x^2) & \cos(x^2) \\ 2x\cos(x^2) & -2x\sin(x^2) \end{vmatrix} = -2x\sin\left(x^2\right) - 2x\cos\left(x^2\right) = -2x.$$

$W(y_1, y_2) = 0$ if and only if $x = 0$. Thus, the set y_1, y_2 is linearly independent on the interval $(0, \infty)$. Since y_1 and y_2 are both solutions of the given differential equation, we conclude that y_1, y_2 form a fundamental set of solutions of the equation on the interval $(0, \infty)$.

(b) **(6 points)** Find a solution of the differential equation in part (a) that satisfies the boundary conditions $y\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{2}$ and $y'\left(-\frac{\sqrt{\pi}}{2}\right) = 0$.

Solution:

The function $y = c_1 \sin(x^2) + c_2 \cos(x^2)$ is a two-parameter family of solutions of the given differential equation. We have

$$y' = 2c_1 x\cos\left(x^2\right) - 2c_2 x\sin\left(x^2\right).$$

Imposing the boundary conditions gives

$$c_1 \sin\frac{\pi}{4} + c_2 \cos\frac{\pi}{4} = \sqrt{2} \ \text{ and } \ 2c_1\left(-\frac{\sqrt{\pi}}{2}\right)\sin\frac{\pi}{4} - 2c_2\left(-\frac{\sqrt{\pi}}{2}\right)\cos\frac{\pi}{4} = 0$$

$\Rightarrow c_1 = c_2 = 1.$

Hence, the required solution is $y = \sin(x^2) + \cos(x^2)$.

Q3. (a) **(8 points)** Verify that $y_{p_1} = xe^x$ and $y_{p_2} = -4x^2$ are, respectively, particular solutions of

$$y'' - 3y' + 4y = e^x(2x-1) \quad \text{and} \quad y'' - 3y' + 4y = -16x^2 + 24x - 8.$$

Solution:

Substituting y_{p_1} to the first equation gives

$$y''_{p_1} - 3y'_{p_1} + 4y_{p_1} = 2e^x + xe^x - 3(e^x + xe^x) + 4xe^x = e^x(2x-1).$$

Thus, y_{p_1} is a particular solution of $y'' - 3y' + 4y = e^x(2x-1)$.

Substituting y_{p_2} to the second equation gives

$$y''_{p_2} - 3y'_{p_2} + 4y_{p_2} = -8 + 24x - 16x^2.$$

Thus, y_{p_2} is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$.

(b) **(5 points)** Use part (a) to find a particular solution of

$$y'' - 3y' + 4y = 2x^2 - 3x + 1 + 4xe^x - 2e^x.$$

Solution: The equation can be written as

$$y'' - 3y' + 4y = 2e^x(2x-1) - \frac{1}{8}(-16x^2 + 24x - 8).$$

By the superposition principle (for nonhomogeneous equations),

$$y_p = 2y_{p_1} - \frac{1}{8}y_{p_2} = 2xe^x + \frac{x^2}{2}$$

is a particular solution of the equation.

Q4. (12 points) Given that $y_1(x) = x + 1$ is a solution of the differential equation

$$(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0,$$

find a second solution $y_2(x)$ of the equation.

Solution: The standard form of the equation is

$$y'' + \frac{2(1+x)}{1-2x-x^2}\, y' - \frac{2}{1-2x-x^2}\, y = 0,$$

$P(x) = \dfrac{2(1+x)}{1-2x-x^2}$.

The second solution is

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{[y_1(x)]^2}\, dx = (x+1) \int \frac{e^{-\int \frac{2(1+x)}{1-2x-x^2}\, dx}}{(x+1)^2}\, dx$$

$$= (x+1) \int \frac{e^{\ln(x^2+2x-1)}}{(x+1)^2}\, dx = (x+1) \int \frac{x^2+2x-1}{(x+1)^2}\, dx$$

$$= (x+1) \int \left[1 - \frac{2}{(x+1)^2}\right] dx = (x+1) \int \left[x + \frac{2}{x+1}\right] = x^2 + x + 2$$

Q5. (12 points) Solve the initial value problem

$$y''' - y'' + 9y' - 9y = 0, \quad y(0) = 13,\ y'(0) = 0,\ y''(0) = 3.$$

Solution: The auxiliary equation is

$$m^3 - m^2 + 9m - 9 = 0 \quad \Rightarrow \quad (m-1)(m^2+9) = 0$$

$$\Rightarrow \quad m_1 = 1,\ m_2 = 3i,\ m_3 = -3i.$$

The general solution is

$$y = c_1 e^x + c_2 \cos 3x + c_3 \sin 3x.$$

We have

$$y' = c_1 e^x - 3c_2 \sin 3x + 3c_3 \cos 3x \quad \text{and} \quad y'' = c_1 e^x - 9c_2 \cos 3x - 9c_3 \sin 3x.$$

Imposing the initial conditions gives

$$y(0) = c_1 + c_2 = 13$$

$$y'(0) = c_1 + 3c_3 = 0$$

$$y''(0) = c_1 - 9c_2 = 3 \quad \Rightarrow \quad c_1 = 12, \quad c_2 = 1, \quad c_3 = -4.$$

The solution is $y = 12e^x + \cos 3x - 4\sin 3x$.

Q6. (15 points) Solve the differential equation

$$y'' - y = x + \cos x$$

by undetermined coefficients (annihilator approach).

Solution: From the auxiliary equation $m^2 - 1 = 0$, we have

$$y_c = c_1 e^x + c_2 e^{-x}.$$

Since $D^2 x = 0$ and $(D^2 + 1)\cos x = 0$, we apply the differential operator $D^2(D^2 + 1)$ to both sides of the equation:

$$D^2(D^2 + 1)(D^2 - 1)y = 0. \tag{1}$$

The auxiliary equation of (1) is

$$m^2(m^2 + 1)(m^2 - 1) = 0$$

$$\Rightarrow \quad m_1 = 1,\ m_2 = -1,\ m_3 = i,\ m_4 = -i,\ m_5 = m_6 = 0.$$

Thus,

$$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + c_5 + c_6 x.$$

After excluding the linear combination of terms corresponding to y_c, we have

$$y_p = A + Bx + C\cos x + D\sin x.$$

Substituting y_p in the given equation gives

$$y_p'' - y_p = -A - Bx - 2C\cos x - 2D\sin x = x + \cos x.$$

Equating coefficients gives

$$A = 0,\ B = -1,\ C = -\frac{1}{2},\ D = 0.$$

We find

$$y_p = -x - \frac{1}{2}\cos x.$$

The general solution is

$$y = c_1 e^x + c_2 e^{-x} - x - \frac{1}{2}\cos x.$$

Q7. (16 points) Solve the differential equation

$$y'' + 8y' + 16y = x^{-2}e^{-4x},\ x > 0.$$

Solution: From the auxiliary equation

$$m^2 + 8m + 16 = (m+4)^2 = 0,$$

we have

$$y_c = c_1 e^{-4x} + c_2 x e^{-4x}.$$

With the identifications $y_1 = e^{-4x}$ and $y_2 = xe^{-4x}$, we next compute the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{-4x} & xe^{-4x} \\ -4e^{-4x} & e^{-4x} - 4xe^{-4x} \end{vmatrix} = e^{-8x} - 4xe^{-8x} + 4xe^{-8x} = e^{-8x}.$$

We identify $f(x) = x^{-2}e^{-4x}$. We obtain

$$W_1 = \begin{vmatrix} 0 & xe^{-4x} \\ x^{-2}e^{-4x} & e^{-4x} - 4xe^{-4x} \end{vmatrix} = -x^{-1}e^{-8x}.$$

$$W_2 = \begin{vmatrix} e^{-4x} & 0 \\ -4e^{-4x} & x^{-2}e^{-4x} \end{vmatrix} = x^{-2}e^{-8x}.$$

and

$$u_1' = -x^{-1} \quad \Rightarrow \quad u_1 = -\ln x$$

$$u_2' = x^{-2} \quad \Rightarrow \quad u_2 = -x^{-1}.$$

Thus,

$$y_p = -e^{-4x}\ln x - e^{-4x}$$

and the general solution is

$$y = y_c + y_p = c_1 e^{-4x} + c_2 x e^{-4x} - e^{-4x}\ln x.$$

Q8. (a) **(6 points)** Find a homogeneous linear differential equation with constant coefficients for which $y = c_1e^{-2x} + c_2e^{6x}$ is the general solution.

Solution: From the general solution we know that the roots of the auxiliary equation are $m_1 = -2,\ m_2 = 6$. This gives the auxiliary equation

$$(m+2)(m-6) = 0 \quad \Rightarrow \quad m^2 - 4m - 12 = 0.$$

Thus, the required equation is

$$y'' - 4y' - 12y = 0.$$

(b) **(7 points)** Use part (a) to find a nonhomogeneous linear differential equation whose general solution is $y = c_1e^{-2x} + c_2e^{6x} + x^2 + 2x$.

Solution: The nonhomogeneous equation is of the form

$$y'' - 4y' - 12y = g(x). \tag{2}$$

Substituting $y_p = x^2 + 2x$ to equation (2) gives

$$\begin{aligned} g(x) &= y_p'' - 4y_p' - 12y_p \\ &= 2 - 4(2x-2) - 12(x^2+2x) \\ &= -12x^2 - 32x - 6. \end{aligned}$$

Thus, the required nonhomogeneous equation is

$$y'' - 4y' - 12y = -12x^2 - 32x - 6.$$