King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

FINAL EXAM – MATH 202 (Term 122)

May 27, 2013

Duration: **180** Minutes

Name : SOLUTION KEY

ID# : ______________________

Section # : ______________________ **Serial #:** __________

Please read the following:

1. Exam has 2 parts: **Part I**: 10 MCQs, **Part II**: 5 Written Questions.
2. Provide **all necessary steps** with **clear writing** for **Part II**.
3. For **Part I (MCQ)**, credit will be given only for the correct answer posted **BELOW.**
4. **Mobiles** and **calculators** are **NOT allowed** in this exam.

Part I: Multiple Choice [**7 Points** for each correct answer]

MCQ #	Student Answer	Points
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
Total Points /**70**		

Part II: Written

Q #	Points	Max
11		14
12		14
13		14
14		16
15		12
Total		**70**

Grand Total /140	

Part I: Multiple Choice

MCQ #	CODE 001	CODE 002	CODE 003	CODE 004
1	D	E	A	C
2	C	D	E	C
3	C	B	A	A
4	B	C	B	E
5	D	D	D	B
6	D	E	B	A
7	B	C	A	B
8	A	B	D	C
9	C	D	B	D
10	E	A	C	A

Part II: Written

Q11. **(14 points)** Find two power series solutions of the differential equation $y'' + x^2y' + xy = 0$ about the ordinary point $x = 0$. Give the first three nonzero terms for each series solution.

Hint: *The assumption* $y = \sum_{n=0}^{\infty} c_n x^n$ *and its first two derivatives lead to*

$$\sum_{k=0}^{\infty}(k+2)(k+1)\ c_{k+2}\ x^k + \sum_{k=2}^{\infty}(k-1)\ c_{k-1}\ x^k + \sum_{k=1}^{\infty}\ c_{k-1}\ x^k = 0. \quad (1)$$

Solution: Equation (1) can be written as

$$2c_2 + 6c_3x + c_0x + \sum_{k=2}^{\infty}(k+2)(k+1)\ c_{k+2}\ x^k + \sum_{k=2}^{\infty}(k-1)\ c_{k-1}\ x^k + \sum_{k=2}^{\infty}\ c_{k-1}\ x^k = 0$$

or $$2c_2 + (c_0 + 6c_3)x + \sum_{k=2}^{\infty}[(k+2)(k+1)\ c_{k+2} + k\ c_{k-1}]\ x^k = 0.$$

We have $c_2 = 0,\ c_3 = -\dfrac{c_0}{6}$ and

$$c_{k+2} = -\frac{kc_{k-1}}{(k+2)(k+1)}, \quad k = 2, 3, 4, \cdots \quad (2)$$

This relation generates consecutive coefficients of the assumed solution as we let k take on the successive integers indicated in (2):

$$k = 2, \qquad c_4 = -\frac{2c_1}{4\cdot 3} = -\frac{c_1}{6}$$

$$k = 3, \qquad c_5 = -\frac{3c_2}{5\cdot 4} = 0$$

$$k = 4, \qquad c_6 = -\frac{4c_3}{6\cdot 5} = \frac{4c_0}{6^2\cdot 5}$$

$$k = 5, \qquad c_7 = -\frac{5c_4}{7\cdot 6} = \frac{5c_1}{6^2\cdot 7}$$

and so on. Now we substitute the coefficients just obtained into the original assumption

$$y = c_0 + c_1x - \frac{c_0}{6}x^3 - \frac{c_1}{6}x^4 + \frac{4c_0}{6^2\cdot 5}x^6 + \frac{5c_1}{6^2\cdot 7}x^7 - \cdots$$
$$= c_0y_1(x) + c_1y_2(x),$$

where

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{4}{6^2\cdot 5}x^6 - \cdots$$

$$y_2(x) = x - \frac{1}{6}x^4 + \frac{5}{6^2\cdot 7}x^7 - \cdots$$

Q12. (14 points) Determine singular points of the differential equation

$$x^2(x^2-1)^2y'' + 2x(x-1)y' + y = 0.$$

Classify each singular point as regular or irregular.

Solution:

It should be clear that $x = 0$, $x = 1$, $x = -1$ are singular points of the equation.

The standard form of the equation is

$$y'' + P(x)y' + Q(x)y = 0,$$

where

$$P(x) = \frac{2}{x(x-1)(x+1)^2} \quad \text{and} \quad Q(x) = \frac{1}{x^2(x+1)^2(x-1)^2}.$$

Since both rational functions

$$p(x) = xP(x) = \frac{2}{(x-1)(x+1)^2} \quad \text{and} \quad q(x) = x^2Q(x) = \frac{1}{(x+1)^2(x-1)^2}$$

are analytic at $x = 0$. We conclude that $x = 0$ is a *regular singular* point.

Similarly, we are led to the conclusion that $x = 1$ is a *regular singular* point. This follows from the fact that

$$p(x) = (x-1)P(x) = \frac{2}{x(x+1)^2} \quad \text{and} \quad q(x) = (x-1)^2Q(x) = \frac{1}{x^2(x+1)^2}$$

are both analytic at $x = 1$.

For $x = -1$,

$$p(x) = (x+1)P(x) = \frac{2}{x(x-1)(x+1)}$$

is not analytic at $x = -1$. Thus, $x = -1$ is an *irregular singular* point.

Q13. (14 points) Solve the initial value problem

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = -2x - y, \quad x(0) = 1,\ y(0) = 1.$$

Solution:

The characteristic equation of the system is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

We find the eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$.

For $\lambda_1 = i$ the system

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} 1-\lambda \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives $k_2 = -(1-i)k_1$. By choosing $k_1 = 1$, we get

$$K_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We form the vectors

$$B_1 = \text{Re}(K_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad B_2 = \text{Im}(K_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right] + c_2\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin t\right].$$

Imposing the initial conditions $x(0) = 1,\ y(0) = 1$ gives $c_1 = 1$ and $c_2 = 2$.

Thus, the required solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t + 2\sin t \\ \cos t - 3\sin t \end{pmatrix}$$

Q14. (16 points) Find the general solution of the system

$$X' = \begin{pmatrix} 3 & 4 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & 3 \end{pmatrix} X.$$

Solution: From the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 & 0 \\ -1 & -\lambda & 2 \\ 0 & 2 & 3-\lambda \end{vmatrix} = \lambda(3-\lambda)^2 = 0,$$

we see that the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$ (of multiplicity two). For $\lambda_1 = 0$ Gauss-Jordan elimination gives

$$(A + 0I|\mathbf{0}) = \left(\begin{array}{ccc|c} 3 & 4 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore $k_1 = \frac{4}{3}k_2, \quad k_3 = -\frac{2}{3}k_2$. The choice $k_2 = -3$ gives an eigenvector

$$K_1 = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}.$$

Similarly, for $\lambda_2 = 3$

$$(A + 3I|\mathbf{0}) = \left(\begin{array}{ccc|c} 0 & 4 & 0 & 0 \\ -1 & -3 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

implies that $k_1 = 2k_3, \quad k_2 = 0$. Choosing $k_3 = 1$ gives an eigenvector

$$K_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

We next solve the system $(A + 3I)P = K_2$:

$$\left(\begin{array}{ccc|c} 0 & 4 & 0 & 2 \\ -1 & -3 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We find $p_1 = 2p_3 - \frac{3}{2}, \quad p_2 = \frac{1}{2}$. Choosing $p_3 = 0$ gives

$$P = \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

The general solution of the given equation is

$$X(t) = c_1 \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_3 \left[\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} te^{3t} + \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} e^{3t} \right]$$

Q15. (12 points) Consider the nonhomogeneous system

$$X' = AX + \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

If the general solution of the associated homogeneous system $X' = AX$ is

$$X_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t,$$

find a particular solution X_p of the nonhomogeneous system.

Solution:

The fundamental matrix of the system is given by

$$\Phi(t) = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix}$$

and

$$\Phi^{-1}(t) = \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}.$$

Here $F(t) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

The required particular solution is

$$\begin{aligned} X_p &= \Phi(t) \int \Phi^{-1}(t) F(t) dt \\ &= \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \int \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} dt \\ &= \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \int \begin{pmatrix} -12 \\ 5e^{-t} \end{pmatrix} dt \\ &= \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \begin{pmatrix} -12t \\ -5e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} -12t - 15 \\ -12t - 10 \end{pmatrix} \end{aligned}$$