King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

EXAM I – MATH 202 (Term 122)

February 26, 2013

Duration: **120** Minutes

Name: Solution key **ID#:** ____________

Section/Instructor: ____________________ **Serial #:** ____________

- Provide **all necessary steps** with **clear writing**.
- **Mobiles** and **calculators** are **NOT allowed** in this exam.

Question #	Marks	Maximum Marks
Q1		13
Q2		7
Q3		10
Q4		16
Q5		15
Q6		10
Q7		17
Q8		12
Total		100

Q1. Consider the differential equation $(y^2 + y) - \dfrac{dy}{dx} = 0$.

(a) **(5 points)** Verify that $y = \dfrac{Ce^x}{1 - Ce^x}$ is a one-parameter family of solutions of the given differential equation.

Solution: $y = \dfrac{Ce^x}{1-Ce^x} \Rightarrow \dfrac{dy}{dx} = \dfrac{Ce^x}{(1-Ce^x)^2}$

$$\Rightarrow \quad y^2 + y = \frac{C^2e^{2x}}{(1-Ce^x)^2} + \frac{Ce^x}{1-Ce^x}$$
$$= \frac{C^2e^{2x} + Ce^x(1-Ce^x)}{(1-Ce^x)^2} = \frac{Ce^x}{(1-Ce^x)^2}$$

Hence,
$$y^2 + y - \frac{dy}{dx} = \frac{Ce^x}{(1-Ce^x)^2} - \frac{Ce^x}{(1-Ce^x)^2} = 0.$$

Therefore, $y = \dfrac{Ce^x}{1-Ce^x}$ is a one-parameter family of solutions of the given DE.

(b) **(4 points)** Find two constant solutions of the given differential equation.

Solution: Constant solutions can be obtained by solving the equation
$y^2 + y = 0$.
We have $y = 0$ or $y = -1$.

(c) **(4 points)** Find a singular solution of the given differential equation.

Solution:
$y = 0$ can be determined from the family in (a) by choosing $C = 0$

$y = -1$ Cannot be obtained from the family.

Thus,

$y = -1$ is a singular solution of the given DE.

Q2. (7 points) Find the values of b such that the initial value problem

$$\frac{dy}{dx} = \sqrt{y - 3x}, \quad y(2) = b$$

has a unique solution.

Solution :

Here $f(x,y) = \sqrt{y-3x}$ and $f_y = \dfrac{1}{2\sqrt{y-3x}}$.

Hence f and f_y are both continuous when $y - 3x > 0 \Rightarrow y > 3x$.

For $x_0 = 2$, we must have $y_0 > 3(2) = 6$.

Therefore $b > 6$.

Q3. (10 points) Solve the separable differential equation

$$(3x + y + xy + 3)dx + (x^2 + 2x)dy = 0.$$

Solution:

The given equation can be written as

$$[3(x+1) + y(x+1)]\,dx + (x^2+2x)\,dy = 0$$

$$\Rightarrow \quad (3+y)(x+1)\,dx + (x^2+2x)\,dy = 0$$

$$\Rightarrow \quad \frac{x+1}{x^2+2x}\,dx + \frac{dy}{3+y} = 0$$

$$\Rightarrow \quad \left(\frac{\frac{1}{2}}{x} + \frac{1/2}{x+2}\right)dx + \frac{dy}{3+y} = 0$$

$$\Rightarrow \quad \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x+2| + \ln|3+y| = C_1$$

$$\Rightarrow \quad (3+y)\sqrt{x(x+2)} = C$$

Q4. Consider the linear differential equation

$$xy' + (1+x)y = e^{-x}\cos^2 x.$$

(a) **(14 points)** Find a solution of the equation that passes through the point $(-\pi, 0)$.

Solution: The standard form of the equation is

$$y' + \left(\frac{1+x}{x}\right)y = \frac{\cos^2 x}{xe^x}.$$

Here $p(x) = \frac{1+x}{x}$. The integrating factor is

$$e^{\int p(x)dx} = e^{\int \frac{1+x}{x}dx} = e^{\int (1+\frac{1}{x})dx} = e^{x+\ln x} = xe^x.$$

We obtain

$$d(xe^x y) = \cos^2 x$$

$$= \frac{1}{2}(1+\cos 2x)$$

$$\Rightarrow xe^x y = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$\Rightarrow y = \frac{1}{2e^x} + \frac{\sin 2x}{4xe^x} + \frac{C}{xe^x}$$

$y(-\pi) = 0$

$$\Rightarrow \left(\frac{1}{2} + \frac{C}{-\pi}\right)e^{\pi} = 0$$

$$\Rightarrow \frac{C}{\pi} = \frac{1}{2} \Rightarrow C = \frac{\pi}{2}$$

So, the solution is

$$y = \frac{1}{2e^x} + \frac{\sin 2x}{4xe^x} + \frac{\pi}{2xe^x}$$

(b) **(2 points)** Give the largest interval in which the solution in (a) is defined.

Solution:

The interval is

$$I = (-\infty, 0)$$

Q5. **(15 points)** Solve the initial value problem

$$ye^{2xy} + x + xe^{2xy}\frac{dy}{dx} = 0, \ y(1) = 0$$

Solution :

The given equation can be written as

$$\left(ye^{2xy} + x\right)dx + xe^{2xy}\,dy = 0$$

Put $M = ye^{2xy} + x$

$N = xe^{2xy}$.

We have $M_y = e^{2xy} + 2xye^{2xy}$

$N_x = e^{2xy} + 2xye^{2xy}$

Since $M_y = N_x$, the equation is an exact equation.

$$\Rightarrow \quad F(x,y) = \int M\,dx + g(y)$$

$$= \int \left(ye^{2xy} + x\right)dx + g(y)$$

$$= \frac{1}{2}e^{2xy} + \frac{1}{2}x^2 + g(y)$$

$$\Rightarrow \quad F_y = N \quad \Rightarrow \quad xe^{2xy} + g'(y) = xe^{2xy}$$

$$\Rightarrow \quad g'(y) = 0$$

put $g(y) = 0$.

The general solution is $\frac{1}{2}e^{2xy} + \frac{1}{2}x^2 = c_1$

$$\Rightarrow \quad e^{2xy} + x^2 = c$$

Imposing the initial condition $y(1) = 0$ gives $c = 2$.

thus, The solution of the ivp is

$$e^{2xy} + x^2 = 2$$

Q6. **(10 points)** Find an integrating factor that makes the differential equation $xy^3 + y + (2x^2y^2 + 2x + 2y^4)y' = 0$ exact.

(Note: **Do not solve** the new equation)

Solution :

The equation can be written as

$$(xy^3+y)\,dx + (2x^2y^2+2x+2y^4)\,dy = 0$$

Put $M = xy^3+y$

$N = 2x^2y^2 + 2x + 2y^4.$

We have $M_y = 3xy^2 + 1$

$N_x = 4xy^2 + 2$

$$\Rightarrow \quad \frac{N_x - M_y}{M} = \frac{(4xy^2+2)-(3xy^2+1)}{xy^3+y}$$

$$= \frac{xy^2+1}{xy^3+y} = \frac{1}{y}$$

The integrating factor is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$$

$$= e^{\int \frac{1}{y}\,dy}$$

$$= e^{\ln y} = y$$

Q7. (a) **(10 points)** Use an appropriate substitution to reduce the differential equation

$$\frac{dy}{dx} = y(xy^3 - 1)$$

to a linear equation. (Note: **Do not solve** the new equation)

Solution: The equation can be written as

$$\frac{dy}{dx} + y = xy^4.$$

It is a Bernoulli equation with $n=4$.

Put $u = y^{1-n} = y^{-3} \iff y = u^{-1/3}$.

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

Substituting $u = y^{-3}$ gives

$$-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} = x u^{-4/3}$$

$$\Rightarrow \frac{du}{dx} - 3u = -3x,$$

(b) **(7 points)** Use a suitable substitution to reduce the differential equation

$$\frac{dy}{dx} = \frac{x+y}{x+y+1}$$

to a separable equation. (Note: **Do not solve** the new equation)

Solution: Put $u = x + y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$$

We obtain

$$\frac{du}{dx} - 1 = \frac{u}{u+1}$$

$$\Rightarrow \frac{du}{dx} = \frac{u}{u+1} + 1 = \frac{2u+1}{u+1}$$

$$\Rightarrow \frac{u+1}{2u+1}\,du = dx$$

Q8. **(12 points)** A thermometer reading $70°F$ is taken from inside a room to the outside, where the air temperature is $20°F$. Four minutes later, the thermometer reads $30°F$. Find the thermometer's reading at $t = 8$ minutes.

Solution: Newton's law of cooling/warming gives

$$\frac{dT}{dt} = k(T - T_m).$$

In this case, $T_m = 20$.

We have $\frac{dT}{dt} = k(T-20)$ or $\frac{dT}{T-20} = k\,dt$

$\Rightarrow \quad T(t) = 20 + ce^{kt}.$

We know that $T(0) = 70$,

$\Rightarrow \quad 70 = 20 + c \Rightarrow c = 50,$

and $T(4) = 30$

$$\left.\begin{aligned} &\Rightarrow 30 = 20 + 50e^{4k} \\ &\Rightarrow e^{4k} = \frac{1}{5} \\ &\Rightarrow k = -\frac{\ln 5}{4}. \end{aligned}\right]$$

Thus,

$$T(t) = 20 + 50e^{-\frac{\ln 5}{4}t}$$

and

$$\begin{aligned} T(8) &= 20 + 50e^{-2\ln 5} \\ &= 20 + 50\left(\tfrac{1}{25}\right) \\ &= 22. \end{aligned}$$

The thermometer's reading after 8 minutes is $22°F$.