Final Version

King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 201
Final Exam – 2012–2013 (122)
Allowed Time: 180 minutes

Name: ____

ID #: __________________________

Section #: _____________ Serial Number: ______________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**

Part I: Written Problems

Question #	Grade	Maximum Points
1		14
2		14
3		12
4		14
5		16
Total:		70

Part II: MCQ Problems

Question #	Answer	Grade	Maximum Points
6	A		07
7	A		07
8	A		07
9	A		07
10	A		07
11	A		07
12	A		07
13	A		07
14	A		07
15	A		07
Total:			70

Q:1 (14 points) Find the critical points of the function

$$f(x,y) = 7x + 4x^2 + y^2 + 2xy^2 + y^4.$$

Classify each point as local maximum, local minimum or saddle point.

Sol: $f_x = 7 + 8x + 2y^2$; $f_{xx} = 8$

$f_y = 2y + 4xy + 4y^3$; $f_{yy} = 2 + 4x + 12y^2$

$f_{xy} = 4y$

$f_y = 0 \Rightarrow y = 0$ or $2y^2 = -2x - 1$

At $y = 0$, $f_x = 0 \Rightarrow x = -\frac{7}{8}$

At $2y^2 = -2x - 1$, $f_x = 0 \Rightarrow 7 + 8x - 2x - 1 = 0 \Rightarrow x = -1$

$y = \pm \frac{1}{\sqrt{2}}$

Critical points are $(-\frac{7}{8}, 0)$, $(-1, -\frac{1}{\sqrt{2}})$, $(-1, \frac{1}{\sqrt{2}})$

At $(-1, \frac{1}{\sqrt{2}})$: $f_{xx} f_{yy} - (f_{xy})^2 = (8)(4) - 8 > 0$

$\therefore (-1, \frac{1}{\sqrt{2}})$ Local minimum

At $(-1, -\frac{1}{\sqrt{2}})$: $f_{xx} f_{yy} - (f_{xy})^2 = 24 > 0$ Local min.

At $(-\frac{7}{8}, 0)$: $f_{xx} f_{yy} - (f_{xy})^2 = (8)(2 - \frac{7}{2}) - 0 = -12 < 0$

$\Rightarrow (-\frac{7}{8}, 0)$ is a saddle point.

Q:2 (14 points) Use Lagrange multipliers to find the maximum and minimum values of

$$f(x,y,z) = x^2y^2z^2$$

subject to the constraint $x^2+y^2+z^2=1$.

Sol. Solve $\nabla f = \lambda \nabla g$ with $x^2+y^2+z^2=1$.

$\nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle$

$\nabla(x^2+y^2+z^2) = \langle 2x, 2y, 2z \rangle$

$\nabla f = \lambda \nabla g \Rightarrow$

$2xy^2z^2 = \lambda 2x$

$2yx^2z^2 = \lambda 2y$

$2zx^2y^2 = \lambda 2z$

If x, y, z are all $\neq 0$, then

$y^2z^2 = x^2z^2 = x^2y^2 = \lambda$

$\Rightarrow x^2 = y^2 = z^2$

Now $x^2+y^2+z^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x^2 = \frac{1}{3}$

So $y^2 = \frac{1}{3}$, $z^2 = \frac{1}{3}$ & $f(x,y,z) = \frac{1}{27}$

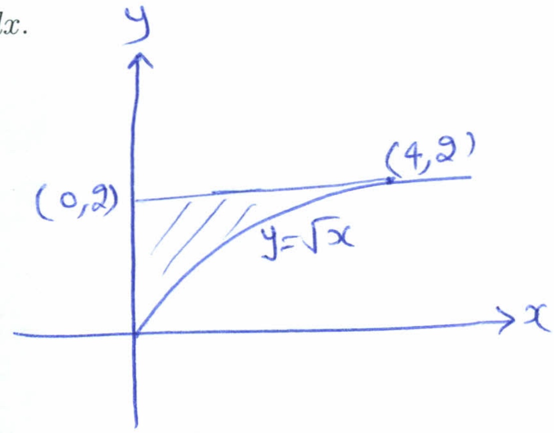
If one of x, y, z is zero, then $f(x,y,z) = 0$

Maximum value $= \frac{1}{27}$

Minimum value $= 0$

Q:3 (12 points) Evaluate the iterated integral

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1} dy dx.$$



Sol.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1} dy\, dx$$

$$= \int_0^2 \int_0^{y^2} \frac{1}{y^3+1} dx\, dy$$

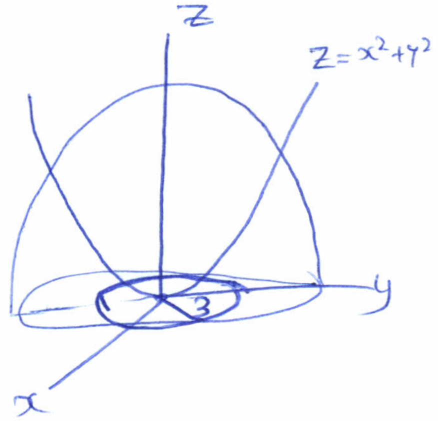
$$= \int_0^2 \frac{1}{y^3+1} \left[x\right]_0^{y^2} dy$$

$$= \int_0^2 \frac{y^2}{y^3+1} dy$$

$$= \frac{1}{3} \ln (y^3+1) \Big|_0^2$$

$$= \frac{1}{3} \left[\ln 9 - \ln 1\right] = \frac{1}{3} \ln 9$$

Q:4 (14 points) Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.



Sol. The projection of the solid in the xy-plane is a circle with radious given by solving the equation

$$x^2 + y^2 = 36 - 3x^2 - 3y^2$$

or $x^2 + y^2 = 9$

$$\text{Volume} = \iiint_E dV$$

$$= \int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 r(36-4r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi} (18r^2 - r^2)\Big|_0^3\,d\theta$$

$$= \int_0^{2\pi} 81\,d\theta$$

$$= 162\pi$$

Q:5 (16 points) Use spherical coordinates to evaluate

$$\iiint_E (x^2+y^2)dV,$$

where E is the region bounded above by the sphere $x^2+y^2+z^2=1$ and below by the cone $z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$

Sol. $x^2+y^2+z^2=1 \iff \rho=1$

$$z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$$

$$\rho\cos\phi=\frac{1}{\sqrt{3}}\rho\sin\phi \quad \text{or} \quad \tan\phi=\sqrt{3}$$

$$\Rightarrow \phi=\frac{\pi}{3}$$

$$\iiint_E (x^2+y^2)\,dV=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/3}\int_0^1 (\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=2\pi\int_0^{\frac{\pi}{3}}\sin^3\phi\,d\phi\int_0^1\rho^4\,d\rho$$

$$=\frac{2\pi}{5}\int_0^{\frac{\pi}{3}}\sin^3\phi\,d\phi$$

$$=\frac{2\pi}{5}\int_0^{\frac{\pi}{3}}(1-\cos^2\phi)\sin\phi\,d\phi$$

$$=\frac{2\pi}{5}\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi/3}$$

$$=\frac{2\pi}{5}\left[-\frac{1}{2}+\frac{1}{24}+1-\frac{1}{3}\right]$$

$$=\frac{2\pi}{5}\cdot\frac{5}{24}=\frac{\pi}{12}$$

Q:6 (7 points) The slope of the tangent line to the polar curve $r = 1 + 2\cos\theta$ at the point $\theta = \frac{\pi}{3}$ is

(A) $\frac{\sqrt{3}}{9}$

(B) $\frac{1}{\sqrt{3}}$

(C) $\frac{1}{3}$

(D) $3\sqrt{3}$

(E) $-\frac{\sqrt{3}}{9}$

Q:7 (7 points) The area of the region that lies inside both curves $r = \cos\theta$ and $r = \sin\theta$ is

(A) $\frac{1}{8}(\pi - 2)$

(B) $\frac{1}{8}(\pi + 2)$

(C) $-\frac{1}{8}(\pi - 2)$

(D) $-\frac{1}{8}(\pi + 2)$

(E) $\frac{1}{8}(2\pi - 2)$

Q:8 (7 points) The area of the surface obtained by rotating the curve parametrized by $x = 3t - t^3,\ y = 3t^2,\ \ 0 \le t \le 1$ about the $x-$ axis is

(A) $\frac{48\pi}{5}$

(B) $\frac{18\pi}{15}$

(C) $-\frac{24\pi}{15}$

(D) $-\frac{16\pi}{15}$

(E) $\frac{\pi}{15}$

Q:9 (7 points) The value of k for which the vectors $\overrightarrow{a} =< 1, 4, -7 >$, $\overrightarrow{b} =< 4, 0, 2 >$ and $\overrightarrow{c} =< k, 0, 1 >$ are coplanar is

(A) 2

(B) -2

(C) 1

(D) -1

(E) 0

Q:10 (7 points) Where does the line that passes through $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x + y + z = 6$?

(A) $(7, -4, 3)$

(B) $(1, 2, 3)$

(C) $(3, 4, -1)$

(D) $(-1, 4, 3)$

(E) $(6, 4, -4)$

Q:11 (7 points) The equation $x^2 - y^2 + z^2 - 4x - 2y - 2z + 4 = 0$ represents

(A) a cone

(B) a hyperboloid of two sheets

(C) a hyperboloid of one sheet

(D) an elliptic paraboloid

(E) a sphere

Q:12 (7 points) Consider the surface

$$x^2z + 3yz^2 + 3xyz = 7.$$

Let $5x + By + Cz = D$ be an equation of the tangent plane to the given surface at $(1, 1, 1)$. The value of $B + C + D$ is

(A) 37

(B) 32

(C) 34

(D) 42

(E) 39

Q:13 (7 points) The maximum rate of change of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(1, 1, 1)$ is

(A) 1

(B) 3

(C) 4

(D) 5

(E) 6

Q:14 (7 points) Consider

$$f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2}, & (x,y) \neq (0,0) \\ 1 \quad , & (x,y) = (0,0) \end{cases}$$

(A) $f(x,y)$ has a removable discontinuity at $(0,0)$

(B) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(C) $f(x,y)$ is continuous at $(0,0)$

(D) $f(x,y)$ is continuous everywhere

(E) $f(x,y)$ is continuous

Q:15 (7 points) If $u = x^4y + y^2z^3$, where $x = rse^t, y = rs^2e^{-t}$ and $z = r^2s$ sint, then the value of $\frac{\partial u}{\partial s}$ when $r = 1, s = 1, t = 0$ is

(A) 6

(B) 30

(C) 4

(D) 25

(E) 10