

King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 201 Major Exam II
The Second Semester of 2012-2013 (122)

Time Allowed: 120 Minutes

Name: _____ ID#: _____

Section/Instructor: _____ Serial #: _____

- Mobiles and calculators are not allowed in this exam.
- Provide all necessary steps required in the solution.

Question #	Marks	Maximum Marks
1		14
2		10
3		14
4		12
5		10
6		12
7		12
8		16
Total		100

Q:1(8 points) Show that the lines with parametric equations

$$L_1 : x = t, y = 1 + 2t, z = 2 + 3t$$

$$L_2 : x = 3 - 4s, y = 2 - 3s, z = 1 + 2s$$

are skew lines.

Sol. Vectors parallel to lines L_1 and L_2 are

$$\vec{v}_1 = \langle 1, 2, 3 \rangle \quad \text{and} \quad \vec{v}_2 = \langle -4, -3, 2 \rangle \quad \text{respectively}$$

The lines are not parallel because $\vec{v}_1 \neq k\vec{v}_2$.

If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$t = 3 - 4s \quad \text{or} \quad t + 4s = 3 \quad (i)$$

$$1 + 2t = 2 - 3s \quad \text{or} \quad 2t + 3s = 1 \quad (ii)$$

$$2 + 3t = 1 + 2s \quad \text{or} \quad 3t - 2s = -1 \quad (iii)$$

Solving (i) and (ii), we get $s = 1, t = -1$.

These values do not satisfy eqn (iii).

Therefore, L_1 and L_2 do not intersect.

Hence L_1 and L_2 are skew lines.

(b)(6 points) Find the distance between the planes

$$z = x + 2y + 1 \quad \text{and} \quad 3x + 6y - 3z = 4.$$

Putting $y = 0, z = 0$ in first plane, we get $(-1, 0, 0)$.

Distance between the plane is the distance from $(-1, 0, 0)$ to the second plane $3x + 6y - 3z = 4$.

Hence

$$D = \frac{|3(-1) + 6(0) - 3(0) - 4|}{\sqrt{3^2 + 6^2 + 3^2}} = \frac{7}{\sqrt{54}}$$

Q:2 (10 points) Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$ and $R(5, 2, 0)$.

Sol. $\vec{PQ} = \langle 2, -4, 4 \rangle$

$$\vec{PR} = \langle 4, -1, -2 \rangle$$

Since \vec{PQ} and \vec{PR} lie in the plane, their cross product $\vec{PQ} \times \vec{PR}$ is orthogonal to the plane

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= \langle 12, 20, 14 \rangle = \vec{n}$$

With the point $P(1, 3, 2)$ and \vec{n} , an eqn of plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

$$12x + 20y + 14z = 100$$

or $6x + 10y + 7z = 50$

Q:3 (14 points) Consider the quadratic surface $4x^2 - 2y^2 + z^2 + 8 = 0$.

- (i) Find the traces of the surface in the planes $y = k$ (k is a constant).
 (ii) Identify and sketch the surface.

(i) Trace in vertical plane $y = k$ is

$$4x^2 + z^2 = 2k^2 - 8$$

or

$$x^2 + \frac{z^2}{4} = \frac{k^2}{2} - 2$$

1) For $|k| > 2$, Ellipse

2) For $|k| = 2$, point

3) For $|k| < 2$, No traces

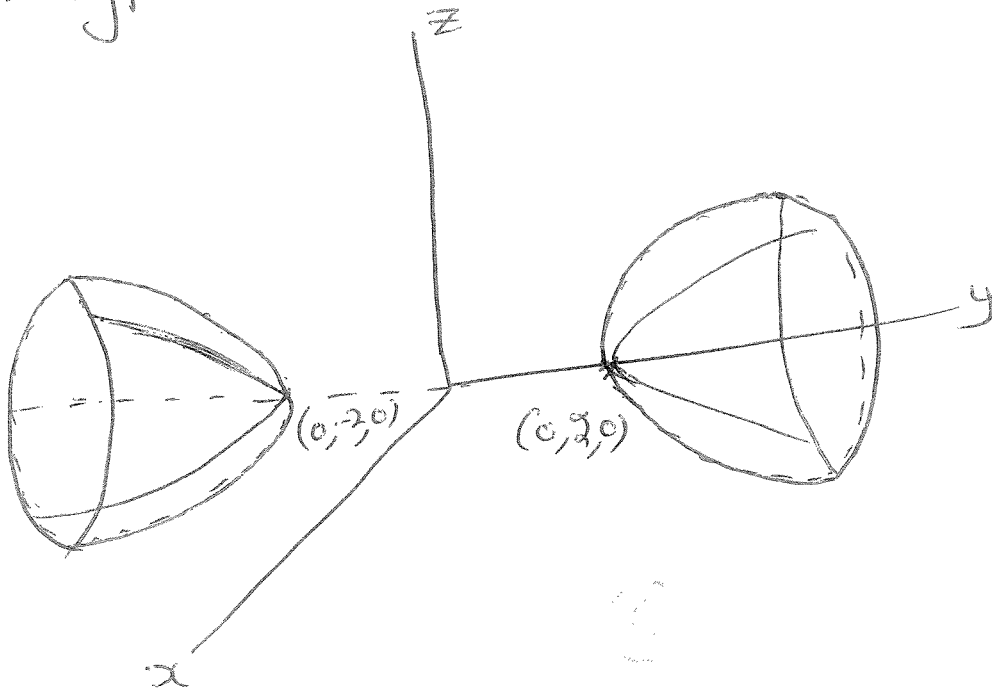
(ii) Dividing by 8, we get

$$\frac{x^2}{2} - \frac{y^2}{4} + \frac{z^2}{8} + 1 = 0$$

or

$$-\frac{x^2}{2} + \frac{y^2}{4} - \frac{z^2}{8} = 1,$$

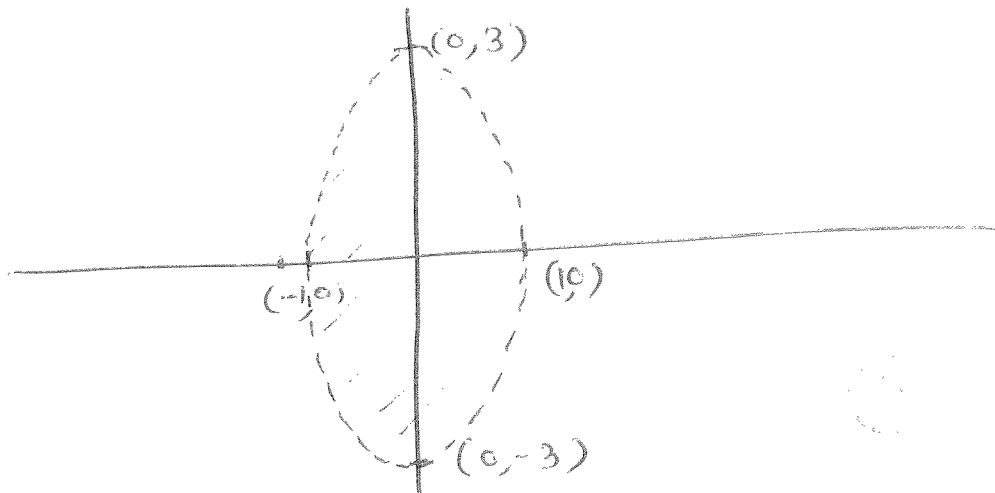
which is hyperboloid of two sheets, the axis is y -axis.



Q:4 Let $f(x, y) = \ln \sqrt{9 - 9x^2 - y^2}$.

(a) (6 points) Find and sketch the domain of f .

$$\begin{aligned} \text{Domain} &= \left\{ (x, y) : 9 - 9x^2 - y^2 > 0 \right\} \\ &= \left\{ (x, y) : \frac{x^2}{1^2} + \frac{y^2}{9} < 1 \right\} \end{aligned}$$



(b) (2 points) Find the range of f

$$R = (-\infty, \ln 3]$$

(c) (4 points) Write an equation of the level curve of f which passes through the point $(\frac{1}{3}, 1)$.

$$f\left(\frac{1}{3}, 1\right) = \ln \sqrt{7}$$

The level curve that passes through $(\frac{1}{3}, 1)$ has an equation

$$\ln \sqrt{7} = \ln \sqrt{9 - 9x^2 - y^2}$$

$$7 = 9 - 9x^2 - y^2$$

$$\text{or } \boxed{9x^2 + y^2 = 2}$$

Q:5 (10 points) Evaluate $\lim_{(x,y) \rightarrow (0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{\sqrt{x^2+y^2}}$.

Let $r^2 = x^2 + y^2$. Then

$$\lim_{r \rightarrow 0^+} \frac{e^{-\frac{1}{r}}}{r} \quad \left(\frac{0}{0}\right)$$

or $\lim_{r \rightarrow 0^+} \frac{\frac{1}{r}}{e^{\frac{1}{r}}} \quad \left(\frac{\infty}{\infty}\right)$

By L, Hospital rule

$$\lim_{r \rightarrow 0^+} \frac{-\frac{1}{r^2}}{\left(-\frac{1}{r^2}\right) e^{\frac{1}{r}}}$$

$$= \lim_{r \rightarrow 0^+} \frac{1}{e^{\frac{1}{r}}} = \frac{1}{\infty} = 0$$

End

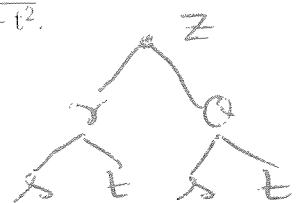
~~$$\lim_{r \rightarrow 0^+} \frac{e^{-\frac{1}{r}}}{r} = \lim_{r \rightarrow 0^+} \frac{\frac{1}{r^2} e^{\frac{1}{r}}}{1}$$

$$= \lim_{r \rightarrow 0^+} \frac{1}{r^2} e^{\frac{1}{r}}$$~~

Q:6 (a)(6 points) Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ when $z = e^{2r} \sin 2\theta$, $r = st - t^2$, $\theta = \sqrt{s^2 + t^2}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r} \frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial s}$$

$$= (2e^{2r} \sin 2\theta) t + (2e^{2r} \cos 2\theta) \frac{s}{\sqrt{s^2 + t^2}}$$



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial t}$$

$$= (2e^{2r} \sin 2\theta) (s - 2t) + (2e^{2r} \cos 2\theta) \frac{t}{\sqrt{s^2 + t^2}}$$

(b)(6 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 1, 0)$ for

$ye^x = 5 \sin 3z + 3z + 1$ by implicit differentiation.

$$F(x, y, z) = ye^x - 5 \sin 3z - 3z - 1$$

$$F_x = ye^x$$

$$F_y = e^x$$

$$F_z = -15 \cos 3z - 3$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{ye^x}{15 \cos 3z + 3}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^x}{15 \cos 3z + 3}$$

$$\frac{\partial z}{\partial x} \Big|_{(0,1,0)} = \frac{1}{3}, \quad \frac{\partial z}{\partial y} \Big|_{(0,1,0)} = \frac{1}{18}$$

$$\frac{1}{18}$$

Q:7 (12 points) Find the linear approximation to the function $f(x,y) = \sqrt{10 - x^2 - 5y^2}$ at $(2,1)$ and use it to approximate $f(2.1, 0.9)$.

$$f(x,y) = \sqrt{10 - x^2 - 5y^2}$$

$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{10 - x^2 - 5y^2}}$$

$$\frac{\partial f}{\partial y} = -\frac{5y}{\sqrt{10 - x^2 - 5y^2}}$$

Linear approximation at $(2,1)$ is

$$L(x,y) \approx f(2,1) + \left. \frac{\partial f}{\partial x} \right|_{(2,1)} (x-2) + \left. \frac{\partial f}{\partial y} \right|_{(2,1)} (y-1)$$

$$= 1 + (-2)(x-2) + (-5)(y-1)$$

$$= 1 - 2x - 5y + 9$$

$$L(2.1, 0.9) \text{ or } f(2.1, 0.9) = 1 - 2 \times 2.1 - 5 \times 0.9 + 9$$

$$= 1 - 4.2 - 4.5 + 9$$

$$= 1.3$$

Q:8 (a) (6 points) Find equation of the tangent plane to the surface $x^2 + y^2 - z^2 = 1$ at a point (a, b, c) .

Normal to the tangent plane is given by

$$\text{grad}(x^2 + y^2 - z^2) / (a, b, c)$$

$$= \langle 2x, 2y, -2z \rangle / (a, b, c)$$

$$= \langle 2a, 2b, -2c \rangle$$

So equation of the tangent plane at (a, b, c) is

$$2a(x-a) + 2b(y-b) - 2c(z-c) = 0.$$

(b) (10 points) Find all points on the hyperboloid $x^2 + y^2 - z^2 = 1$ at which the tangent plane is parallel to the plane $x + y + z = 1$.

Normal to the plane $x + y + z = 1$ is $\langle 1, 1, 1 \rangle$

Points (a, b, c) on the hyperboloid at which the tangent plane is parallel to the plane $x + y + z = 1$ are

$$\langle 2a, 2b, -2c \rangle \parallel \langle 1, 1, 1 \rangle$$

Or

$$\langle 2a, 2b, -2c \rangle = k \langle 1, 1, 1 \rangle$$

$$\Rightarrow a = \frac{k}{2}, \quad b = \frac{k}{2}, \quad c = -\frac{k}{2}$$

$$a^2 + b^2 - c^2 = 1 \Rightarrow \left(\frac{k}{2}\right)^2 + \left(\frac{k}{2}\right)^2 - \left(-\frac{k}{2}\right)^2 = 1$$

$$\Rightarrow k = \pm 2$$

pts

$$(1, 1, -1), \quad (-1, -1, 1).$$