

King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 201 Major Exam I
The Second Semester of 2012-2013 (122)

Time Allowed: 120 Minutes

Name: _____ ID#: _____

Section/Instructor: _____ Serial #: _____

- Mobiles and calculators are not allowed in this exam.
 - Provide all necessary steps required in the solution.
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Question #	Marks	Maximum Marks
1		12
2		12
3		12
4		14
5		14
6		10
7		14
8		12
Total		100

Q:1 Consider the parametric equations $x = 4 \sin t$, $y = -5 \cos t$.

(a) (4 points) Eliminate the parameter to find a cartesian equation.

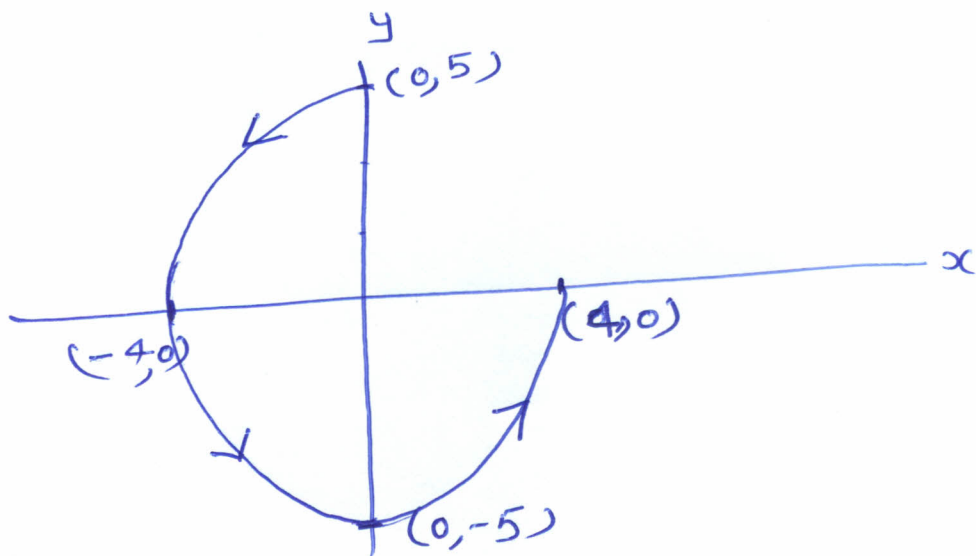
(b) (8 points) Sketch the curve for $-\pi \leq t \leq \frac{\pi}{2}$ and mark the direction in which the curve is traced as t increases.

(a) $\frac{x}{4} = \sin t$ and $\frac{y}{-5} = \cos t$.

Then $\frac{x^2}{16} + \frac{y^2}{25} = \sin^2 t + \cos^2 t = 1$

(b)

t	x	y
$-\pi$	0	5
$-\frac{\pi}{2}$	-4	0
0	0	-5
$\frac{\pi}{2}$	4	0



Q:2 (a)(8 points) At what point(s) on the curve $x = t^2 + 4t$, $y = 6t^2$ is the tangent parallel to the line with parametric equations $x = -3t$, $y = 12t - 5$?

Sol. $x = t^2 + 4t$
 $\frac{dx}{dt} = 2t + 4$

$y = 6t^2$
 $\frac{dy}{dt} = 12t$

The curve has slope $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t}{2t+4}$ (i)

The line with parametric equations
 $x = -3t$, $y = 12t - 5$ is

$y = 12\left(-\frac{x}{3}\right) - 5 = -4x - 5$

Slope of line is -4 . (ii)

From (i) and (ii), we have

$\frac{12t}{2t+4} = -4$

or $\frac{3t}{2t+4} = -1$

$\Rightarrow 5t + 4 = 0 \Rightarrow t = -\frac{4}{5}$

Point on the curve is $(x, y)_t = \left(-\frac{64}{25}, \frac{96}{25}\right)$.

(b) (4 points) At what point(s) on the curve of part (a) is the tangent vertical?

Sol. $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$2t + 4 = 0$

$\Rightarrow t = -2$

Point is $(0, 24)$.

Q:3 (12 points) Find the length of the curve

$$x = 8 \cos t + 8t \sin t, \quad y = 8 \sin t - 8t \cos t; \quad 0 \leq t \leq \frac{\pi}{2}.$$

Sol.

$$\begin{aligned} \frac{dx}{dt} &= -8 \sin t + 8 \sin t + 8t \cos t \\ &= 8t \cos t \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= 8 \cos t - 8 \cos t + 8t \sin t \\ &= 8t \sin t \end{aligned}$$

$$\text{Length} = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^{\pi/2} \sqrt{64t^2} dt$$

$$= \int_0^{\pi/2} 8t dt$$

$$= 4 \left(t^2\right)_0^{\pi/2}$$

$$= \pi^2$$

Q:4 (a)(8 points) Write the polar equation $r = -8 \cos \theta$ in cartesian coordinates.

Sol:

$$r = -8 \cos \theta$$

$$r^2 = -8r \cos \theta$$

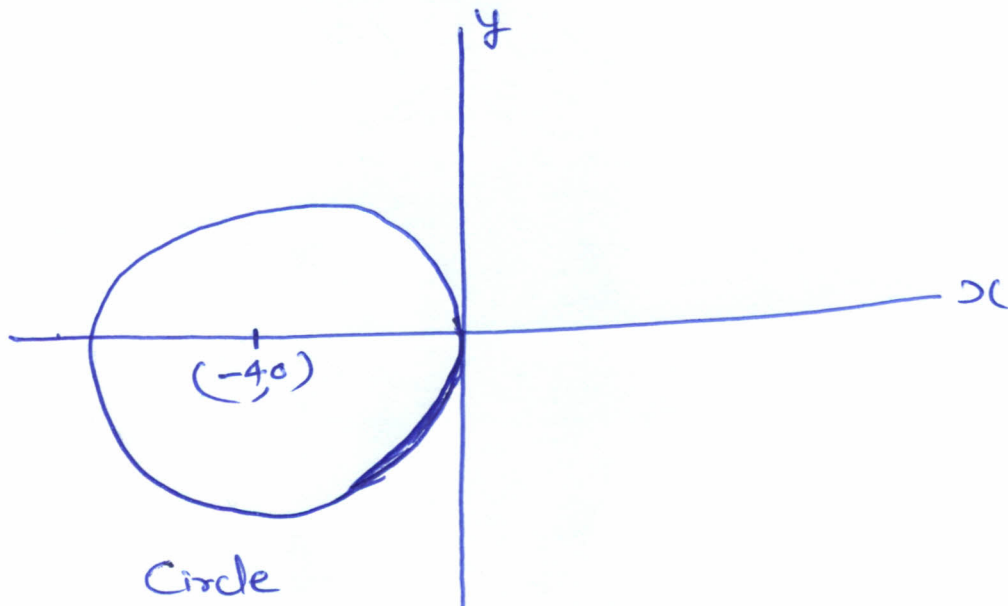
$$x^2 + y^2 = -8x$$

$$\Rightarrow x^2 + y^2 + 8x = 0$$

$$\Rightarrow x^2 + 8x + 16 + y^2 = 16$$

$$\Rightarrow (x+4)^2 + (y-0)^2 = 4^2.$$

(b) (6 points) Sketch the graph of the resulting equation in Part(a).



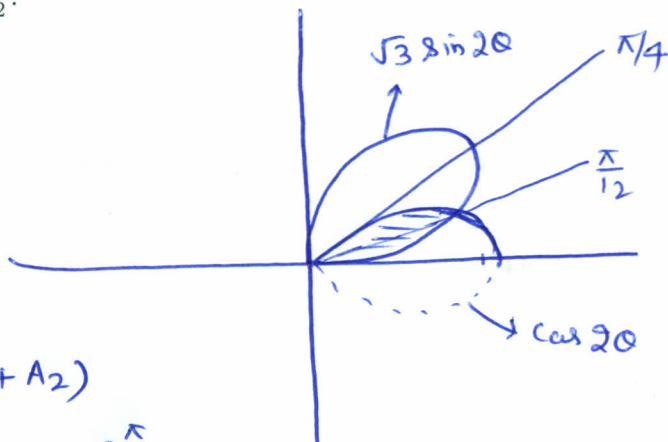
Q:5 (14 points) Find the area of the region that lies inside both curves $r = \cos 2\theta$ and $r = \sqrt{3} \sin 2\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$.

Sol. The curves intersect at

$$\sqrt{3} \sin 2\theta = \cos 2\theta$$

$$\tan 2\theta = \frac{1}{\sqrt{3}}$$

$$2\theta = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{12}$$



Area of the region = $(A_1 + A_2)$

$$\text{where } A_1 = \int_0^{\frac{\pi}{12}} \frac{1}{2} r^2 d\theta = \frac{3}{2} \int_0^{\frac{\pi}{12}} \sin^2 2\theta d\theta$$

$$= \frac{3}{4} \int_0^{\frac{\pi}{12}} (1 - \cos 4\theta) d\theta$$

$$= \frac{3}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\frac{\pi}{12}} = \frac{3}{4} \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8} \right)$$

$$A_2 = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{1}{2} (\cos 2\theta)^2 d\theta = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \left(\frac{1 + \cos 4\theta}{4} \right) d\theta$$

$$= \frac{1}{4} \left[\theta + \frac{\sin 4\theta}{4} \right]_{\frac{\pi}{12}}^{\frac{\pi}{4}} = \frac{1}{4} \left[\frac{\pi}{4} - \frac{\pi}{12} - \frac{\sqrt{3}}{8} \right]$$

$$= \frac{\pi}{24} - \frac{\sqrt{3}}{32}$$

$$\text{Area} = \frac{3\pi}{48} + \frac{\pi}{24} - \frac{4\sqrt{3}}{32}$$

$$= \frac{5\pi}{48} - \frac{4\sqrt{3}}{32}$$

Q:6 (10 points) Find an equation of the sphere that passes through the point $(2, -4, 3)$ and has center $(1, 2, 5)$. Describe the intersection of this sphere with the xz -plane.

Sol. Radius = $\sqrt{(1-2)^2 + (2+4)^2 + (5-3)^2}$
 $= \sqrt{1+36+4} = \sqrt{41}$

Equation of sphere is

$$(x-1)^2 + (y-2)^2 + (z-5)^2 = 41$$

The intersection of this sphere with xz -plane is the set of all points on the sphere whose y -coordinate is zero.

Putting $y=0$ in the eqn of sphere, we get

$$(x-1)^2 + (z-5)^2 = 37,$$

which represents a circle in the xz -plane with centre $(1, 0, 5)$ and radius $\sqrt{37}$.

Q:7 (8 points) Let $\vec{a} = \langle 1, 1, 1 \rangle$ and $\vec{b} = \langle 2, 3, 4 \rangle$. Find $\text{comp}_{\vec{a}} \vec{b}$ and $\text{proj}_{\vec{a}} \vec{b}$.

Sol.
$$\text{Comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$= \frac{1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4}{\sqrt{1+1+1}} = \frac{9}{\sqrt{3}}$$

$$\text{Proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \left(\frac{\vec{a}}{|\vec{a}|} \right)$$

$$= \frac{9}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$

$$= \langle 3, 3, 3 \rangle$$

(b) (6 points) Show that, in general, if \vec{u} , \vec{v} are non-zero vectors, then $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$ is orthogonal to \vec{v} .

Sol.

$$\begin{aligned} & (\vec{u} - \text{proj}_{\vec{v}} \vec{u}) \cdot \vec{v} \\ &= \vec{u} \cdot \vec{v} - \text{proj}_{\vec{v}} \vec{u} \cdot \vec{v} \\ &= \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} (\vec{v} \cdot \vec{v}) \\ &= \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} \\ &= 0 \end{aligned}$$

$\Rightarrow \vec{u} - \text{proj}_{\vec{v}} \vec{u}$ is orthogonal to \vec{v} .

Q:8 (12 points) If the points $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$, $D(0, 1, k)$ are in the same plane, then find the value of k .

$$\vec{AB} = \langle -1, 2, 0 \rangle$$

$$\vec{AC} = \langle -1, 0, 3 \rangle$$

$$\vec{AD} = \langle -1, 1, k \rangle$$

The volume of the parallelepiped formed from these vectors must be zero. Then

$$\begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \\ -1 & 1 & k \end{vmatrix} = 0$$

$$-1(-3) - 2(-k+3) = 0$$

$$3 + 2k - 6 = 0$$

$$2k = 3$$

$$\boxed{k = \frac{3}{2}}$$