Q:1(8 points) Show that the lines with parametric equations

$$L_1 : x = t, y = 1 + 2t, z = 2 + 3t$$

$$L_2 : x = 3 - 4s, y = 2 - 3s, z = 1 + 2s$$

are skew lines.

Sol. Vectors parallel to lines L_1 and L_2 are $\vec{V_1} = \langle 1,2,3 \rangle$ and $\vec{V_2} = \langle -4,-3,2 \rangle$ respectively. 2

The lines are not parallel because $\vec{V_1} \neq k\vec{V_2}$. 2

If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$t = 3 - 4s$		$t + 4s = 3$	(i)
$1 + 2t = 2 - 3s$	Or	$2t + 3s = 1$	(ii) 2
$2 + 3t = 1 + 2s$		$3t - 2s = -1$	(iii)

Solving (i) and (ii), we get $s = 1, t = -1$. 1

These values do not satisfy eqn (iii).

Therefore, L_1 and L_2 do not intersect.

Hence L_1 and L_2 are skew lines. 1

(b)(6 points) Find the distance between the planes

$$z = x + 2y + 1 \quad \textit{and} \quad 3x + 6y - 3z = 4.$$

Putting $y=0$, $z=0$ in first plane, we get $(-1,0,0)$. 2

Distance between the plane is the distance from $(-1,0,0)$ to the second plane $3x + 6y - 3z = 4$.

Hence

$$D = \frac{|3(-1) + 6(0) - 3(0) - 4|}{\sqrt{3^2 + 6^2 + 3^2}} = \frac{7}{\sqrt{54}}$$ 4

Q:2 (10 points) Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$.

Sol. $\overrightarrow{PQ} = \langle 2, -4, 4 \rangle$ 2

$\overrightarrow{PR} = \langle 4, -1, -2 \rangle$

Since $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie in the plane, their cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$ is orthogonal to the plane

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$ 1

$$= \langle 12, 20, 14 \rangle = \vec{n}$$ 3

With the point $P(1,3,2)$ and $\vec{n}$, an eqn of plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$ 2

$$12x + 20y + 14z = 100$$

or $$6x + 10y + 7z = 50$$

Q:3 (14 points) Consider the quadratic surface $4x^2 - 2y^2 + z^2 + 8 = 0$.

(i) Find the traces of the surface in the planes y = k (k is a constant).

(ii) Identify and sketch the surface.

(i) Trace in vertical plane $y = k$ is

$$4x^2 + z^2 = 2k^2 - 8$$

2 or

$$x^2 + \frac{z^2}{4} = \frac{k^2}{2} - 2$$

1 For $|k| > 2$, Ellipse

1 For $|k| = 2$, point

1 For $|k| < 2$, No traces

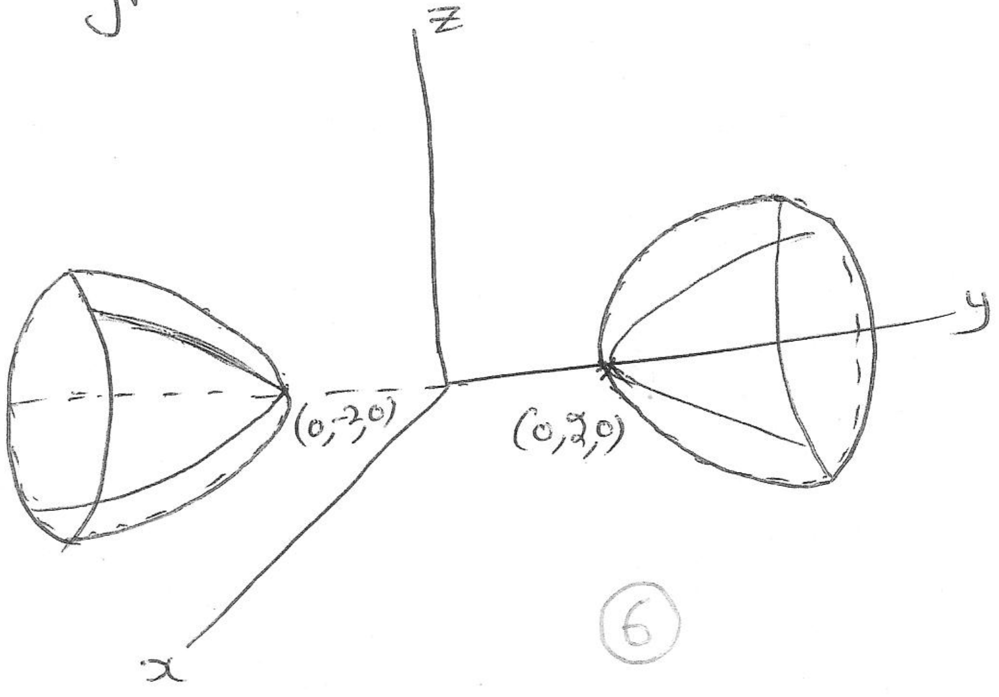
(ii) Dividing by 8, we get

$$\frac{x^2}{2} - \frac{y^2}{4} + \frac{z^2}{8} + 1 = 0$$

or

1 $$-\frac{x^2}{2} + \frac{y^2}{4} - \frac{z^2}{8} = 1,$$

2 which is hyperboloid of two sheets, the axis is y-axis.



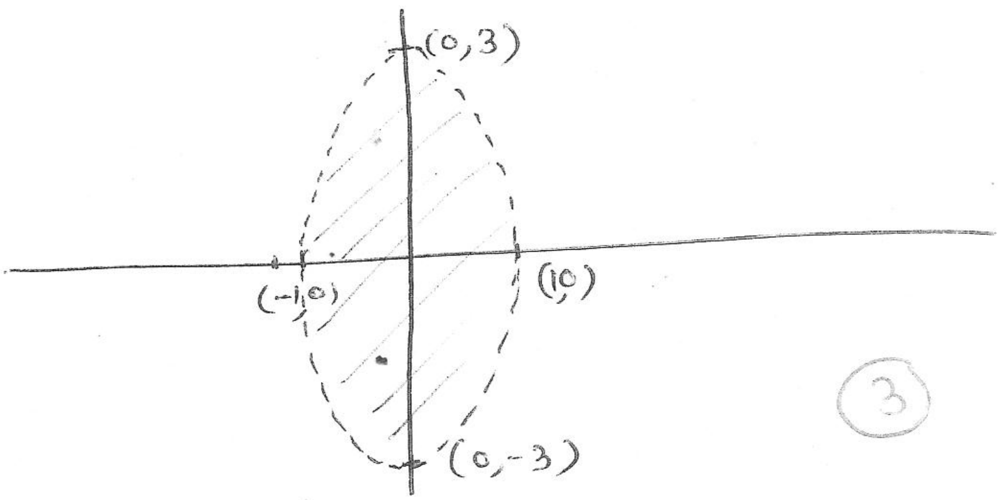
6

Q:4 Let $f(x,y) = \ln\sqrt{9-9x^2-y^2}$.

(a)(6 points) Find and sketch the domain of f.

$$\text{Domain} = \{(x,y) : 9 - 9x^2 - y^2 > 0\}$$
$$= \left\{(x,y) : \frac{x^2}{1^2} + \frac{y^2}{3^2} < 1\right\}$$

3



3

(b) (2 points) Find the range of f

$$R = (-\infty, \ln 3]$$

2

(c) (4 points) Write an equation of the level curve of f which passes through the point $(\frac{1}{3}, 1)$.

$$f(\tfrac{1}{3}, 1) = \ln\sqrt{7}$$

2

The level curve that passes through $(\frac{1}{3}, 1)$ has an equ^n

$$\ln\sqrt{7} = \ln\sqrt{9-9x^2-y^2}$$

2

$$7 = 9 - 9x^2 - y^2 \quad \text{or} \quad \boxed{9x^2 + y^2 = 2}$$

Q:5 (10 points) Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{\sqrt{x^2+y^2}}$.

Let $r^2 = x^2+y^2$. Then (2)

$$\lim_{r\to 0^+} \frac{e^{-\frac{1}{r}}}{r} \quad \left(\frac{0}{0}\right) \quad (2)$$

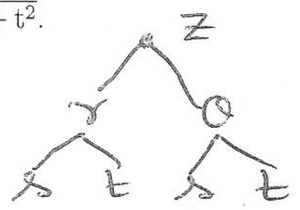
or $$\lim_{r\to 0^+} \frac{\frac{1}{r}}{e^{\frac{1}{r}}} \quad \left(\frac{\infty}{\infty}\right) \quad (2)$$

By L, Hospital rule

$$\lim_{r\to 0^+} \frac{-\frac{1}{r^2}}{\left(-\frac{1}{r^2}\right) e^{\frac{1}{r}}} \quad (2)$$

$$= \lim_{r\to 0^+} \frac{1}{e^{\frac{1}{r}}} = \frac{1}{\infty} = 0 \quad (2)$$

Q:6 (a)(6 points) Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ when $z = e^{2r}\sin 2\theta$, $r = st - t^2$, $\theta = \sqrt{s^2 + t^2}$.



1 $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

2 $$= \left(2e^{2r}\sin 2\theta\right)t + \left(2e^{2r}\cos 2\theta\right)\cdot\frac{s}{\sqrt{s^2+t^2}}$$

1 $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

2 $$= \left(2e^{2r}\sin 2\theta\right)(s-2t) + \left(2e^{2r}\cos 2\theta\right)\frac{t}{\sqrt{s^2+t^2}}$$

(b)(6 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0,1,0)$ for

$ye^x = 5\sin 3z + 3z + 1$ by implicit differentiation.

$$F(x,y,z) = ye^x - 5\sin 3z - 3z - 1$$

$$F_x = ye^x$$

$$F_y = e^x$$ 1

$$F_z = -15\cos 3z - 3$$ 1

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{ye^x}{15\cos 3z + 3}$$ 1

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^x}{15\cos 3z + 3}$$ 1

$$\left.\frac{\partial z}{\partial x}\right|_{(0,1,0)} = \frac{1}{3}, \quad \left.\frac{\partial z}{\partial y}\right|_{(0,1,0)} = \frac{1}{18}$$

1 1

Q:7 (12 points) Find the linear approximation to the function $f(x,y) = \sqrt{10 - x^2 - 5y^2}$ at $(2,1)$ and use it to approximate $f(2.1, 0.9)$.

$$f(x,y) = \sqrt{10 - x^2 - 5y^2}$$

$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{10 - x^2 - 5y^2}} \quad 2$$

$$\frac{\partial f}{\partial y} = -\frac{5y}{\sqrt{10 - x^2 - 5y^2}} \quad 2$$

Linear approximation at $(2,1)$ is

$$L(x,y) \approx f(2,1) + \left.\frac{\partial f}{\partial x}\right|_{(2,1)} (x-2) + \left.\frac{\partial f}{\partial y}\right|_{(2,1)} (y-1) \quad 2$$

$$= 1 + (-2)(x-2) + (-5)(y-1) \quad 1 + 1 + 1$$

$$= 1 - 2x - 5y + 9$$

$$L(2.1, 0.9) \text{ or } f(2.1, 0.9) = 1 - 2 \times 2.1 - 5 \times 0.9 + 9 \quad 2$$

$$= 1 - 4.2 - 4.5 + 9$$

$$= 1.3 \quad 1$$

Q:8 (a)(6 points) Find equation of the tangent plane to the surface $x^2+y^2-z^2=1$ at a point (a,b,c).

Normal to the tangent plane is given by

$$\text{grad}\,(x^2+y^2-z^2)\Big|_{(a,b,c)}$$

$$= \langle 2x, 2y, -2z\rangle\Big|_{(a,b,c)} \quad 3$$

$$= \langle 2a, 2b, -2c\rangle. \quad 1$$

So equation of the tangent plane at (a,b,c) is

$$2a(x-a)+2b(y-b)-2c(z-c)=0. \quad 2$$

(b) (10 points) Find all points on the hyperboloid $x^2+y^2-z^2=1$ at which the tangent plane is parallel to the plane $x+y+z=1$.

Normal to the plane $x+y+z=1$ is $\langle 1,1,1\rangle$ 3

Points (a,b,c) on the hyperbaloid at which the tangent plane is parallel to the plane $x+y+z=1$ are

$$\langle 2a, 2b, -2c\rangle \parallel \langle 1,1,1\rangle$$

Or

$$\langle 2a, 2b, -2c\rangle = k\langle 1,1,1\rangle \quad 2$$

$$\Rightarrow a=\frac{k}{2},\; b=\frac{k}{2},\; c=-\frac{k}{2}. \quad 3$$

$$a^2+b^2+c^2=1 \Rightarrow \left(\tfrac{k}{2}\right)^2+\left(\tfrac{k}{2}\right)^2-\left(\tfrac{k}{2}\right)^2=1$$

$$\Rightarrow k=\pm 2 \quad 1$$

pts $(1,1,1)$, $(-1,-1,1)$. 1