King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 201 Major Exam I
The Second Semester of 2012-2013 (122)

Time Allowed: 120 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- Mobiles and calculators are not allowed in this exam.
- Provide all necessary steps required in the solution.

Question #	Marks	Maximum Marks
1		12
2		12
3		12
4		14
5		14
6		10
7		14
8		12
Total		100

Q:1 Consider the parametric equations $x = 4\sin t,\ y = -5\cos t$.

(a) (4 points) Eliminate the parameter to find a cartesian equation.

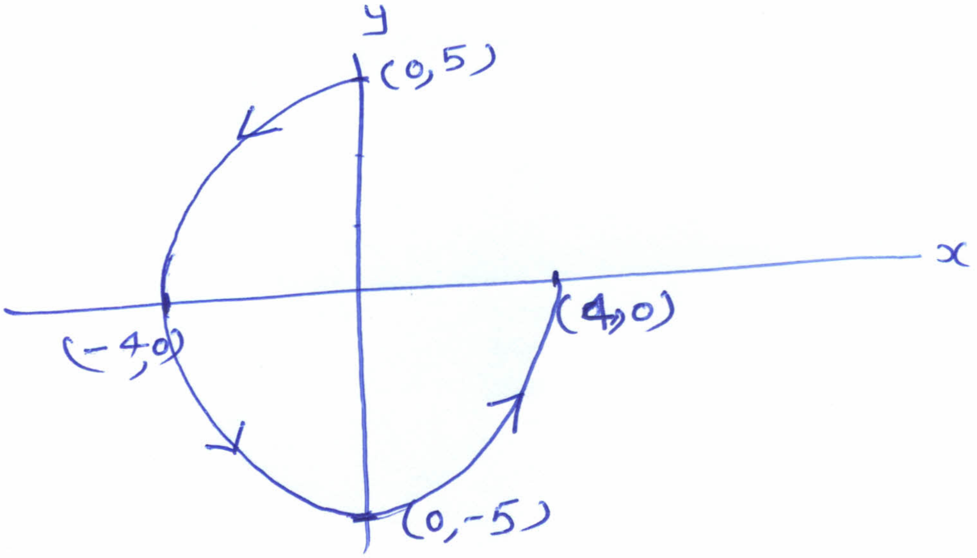
(b) (8 points) Sketch the curve for $-\pi \le t \le \frac{\pi}{2}$ and mark the direction in which the curve is traced as t increases.

(a) $\frac{x}{4} = \sin t$ and $\frac{y}{-5} = \cos t$.

Then $\frac{x^2}{16} + \frac{y^2}{25} = \sin^2 t + \cos^2 t = 1$

(b)

t	x	y
$-\pi$	0	5
$-\frac{\pi}{2}$	-4	0
0	0	-5
$\frac{\pi}{2}$	4	0



Q:2 (a)(8 points) At what point(s) on the curve $x = t^2 + 4t,\ y = 6t^2$ is the tangent parallel to the line with parametric equations $x = -3t,\ y = 12t - 5$?

Sol. $x = t^2 + 4t$ $\qquad$ $y = 6t^2$

$\frac{dx}{dt} = 2t + 4$ $\qquad$ $\frac{dy}{dt} = 12t$

The curve has slope $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t}{2t+4}$ $\quad$ (i)

The line with parametric equations

$x = -3t,\ y = 12t - 5$ is

$y = 12\left(-\frac{x}{3}\right) - 5 = -4x - 5$

Slope of line is -4. $\quad$ (ii)

From (i) and (ii), we have

$\frac{12t}{2t+4} = -4$ or $\frac{3t}{2t+4} = -1$

$\Rightarrow 5t + 4 = 0 \Rightarrow t = -\frac{4}{5}$

Point on the curve is $(x, y)_t = \left(-\frac{64}{25}, \frac{96}{25}\right)$.

(b) (4 points) At what point(s) on the curve of part (a) is the tangent vertical ?

Sol. $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$2t + 4 = 0$

$\Rightarrow t = -2$

Point is $(0, 24)$.

Q:3 (12 points) Find the length of the curve

$$x = 8\cos t + 8t\ \sin t, \qquad y = 8\sin t - 8t\cos t; \qquad 0 \le t \le \frac{\pi}{2}.$$

Sol.
$$\frac{dx}{dt} = -8\sin t + 8\sin t + 8t\cos t$$
$$= 8t\cos t$$
$$\frac{dy}{dt} = 8\cos t - 8\cos t + 8t\sin t$$
$$= 8t\sin t$$

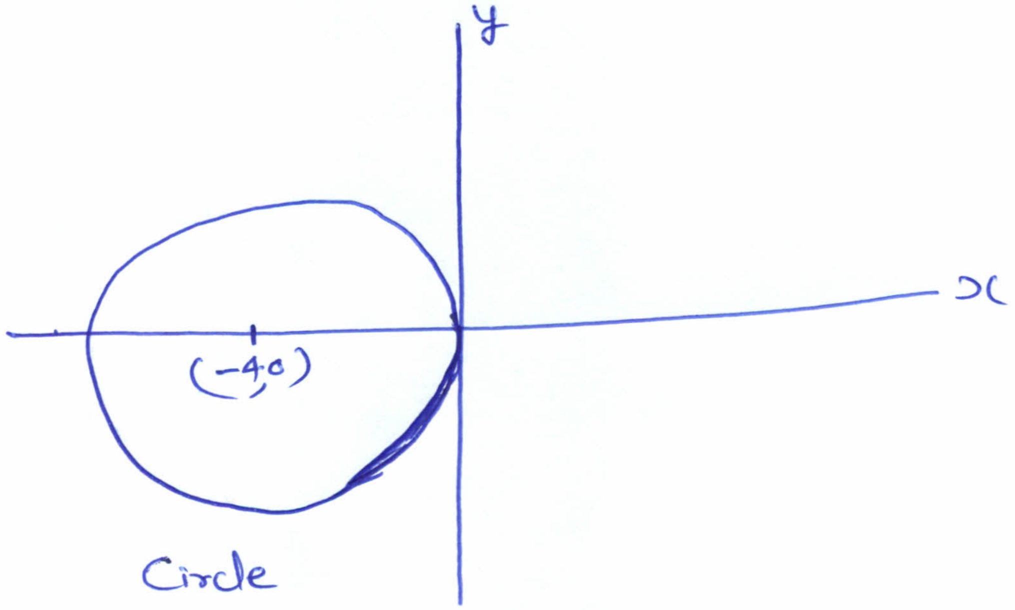
$$\text{Length} = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{64t^2}\ dt$$
$$= \int_0^{\pi/2} 8t\ dt$$
$$= 4\left(t^2\right)_0^{\pi/2}$$
$$= \pi^2$$

Q:4 (a)(8 points) Write the polar equation $r = -8\ \cos\theta$ in cartesian coordinates.

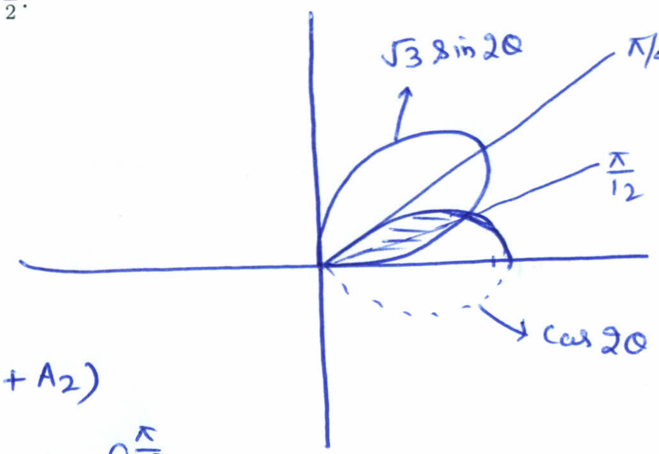
Sol.

$$r = -8\cos\theta$$
$$r^2 = -8r\cos\theta$$
$$x^2+y^2 = -8x$$
$$\Rightarrow x^2+y^2+8x=0$$
$$\Rightarrow x^2+8x+16+y^2=16$$
$$\Rightarrow (x+4)^2+(y-0)^2=4^2.$$

(b) (6 points) Sketch the graph of the resulting equation in Part(a).



Q:5 (14 points) Find the area of the region that lies inside both curves $r = \cos 2\theta$ and $r = \sqrt{3}\ \sin 2\theta$ for $0 \le \theta \le \frac{\pi}{2}$.



Sol. The curve intersect at

$$\sqrt{3}\,\sin 2\theta = \cos 2\theta$$

$$\tan 2\theta = \frac{1}{\sqrt{3}}$$

$$2\theta = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{12}$$

Area of the region $= (A_1 + A_2)$

where $A_1 = \displaystyle\int_0^{\frac{\pi}{12}} \frac{1}{2} r^2\, d\theta = \frac{3}{2}\int_0^{\frac{\pi}{12}} \sin^2 2\theta\, d\theta$

$$= \frac{3}{4}\int_0^{\frac{\pi}{12}} (1 - \cos 4\theta)\, d\theta$$

$$= \frac{3}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{12}} = \frac{3}{4}\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)$$

$$A_2 = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{1}{2}(\cos 2\theta)^2\, d\theta = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \left(\frac{1 + \cos 4\theta}{4}\right) d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}} = \frac{1}{4}\left[\frac{\pi}{4} - \frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

$$= \frac{\pi}{24} - \frac{\sqrt{3}}{32}$$

$$\text{Area} = \frac{3\pi}{48} + \frac{\pi}{24} - \frac{4\sqrt{3}}{32}$$

$$= \frac{5\pi}{48} - \frac{4\sqrt{3}}{32}$$

Q:6 (10 points) Find an equation of the sphere that passes through the point $(2, -4, 3)$ and has center $(1, 2, 5)$. Describe the intersection of this sphere with the xz-plane.

Sol. Radious $= \sqrt{(1-2)^2 + (2+4)^2 + (5-3)^2}$

$= \sqrt{1+36+4} = \sqrt{41}$

Equation of sphere is

$$(x-1)^2 + (y-2)^2 + (z-5)^2 = 41$$

The intersection of this sphere with xz-plane is the set of all points on the sphere whose y-coordinate is zero.

Putting $y=0$ in the eqn of sphere, we get

$$(x-1)^2 + (z-5)^2 = 37,$$

which represents a circle in the xz-plane with centre $(1, 0, 5)$ and radious $\sqrt{37}$.

Q:7 (8 points) Let $\overrightarrow{a} =< 1, 1, 1 >$ and $\overrightarrow{b} =< 2, 3, 4 >$. Find $\text{comp}_{\overrightarrow{a}}\overrightarrow{b}$ and $\text{proj}_{\overrightarrow{a}}\overrightarrow{b}$.

Sol.

$$\text{Comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$= \frac{1\cdot 2 + 1\cdot 3 + 1\cdot 4}{\sqrt{1+1+1}} = \frac{9}{\sqrt{3}}$$

$$\text{Proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\left(\frac{\vec{a}}{|\vec{a}|}\right)$$

$$= \frac{9}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\langle 1,1,1\rangle$$

$$= \langle 3, 3, 3\rangle.$$

(b) (6 points) Show that, in general, if $\overrightarrow{u}$, $\overrightarrow{v}$ are non-zero vectors, then $\overrightarrow{u} - \text{proj}_{\overrightarrow{v}}\overrightarrow{u}$ is orthogonal to $\overrightarrow{v}$.

Sol.

$$\left(\vec{u} - \text{proj}_{\vec{v}}\vec{u}\right)\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{v} - \text{proj}_{\vec{v}}\vec{u}\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}(\vec{v}\cdot\vec{v})$$

$$= \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v}$$

$$= 0$$

$\Rightarrow \vec{u} - \text{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$.

Q:8 (12 points) If the points $A(1,0,0)$, $B(0,2,0)$, $C(0,0,3)$, $D(0,1,k)$ are in the same plane, then find the value of k.

$$\vec{AB} = \langle -1, 2, 0 \rangle$$

$$\vec{AC} = \langle -1, 0, 3 \rangle$$

$$\vec{AD} = \langle -1, 1, k \rangle$$

The volume of the parallelopiped formed from these vectors must be zero. Then

$$\begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \\ -1 & 1 & k \end{vmatrix} = 0$$

$$-1(-3) - 2(-k+3) = 0$$

$$3 + 2k - 6 = 0$$

$$2k = 3$$

$$\boxed{k = \frac{3}{2}}$$