Quiz 5 (a)

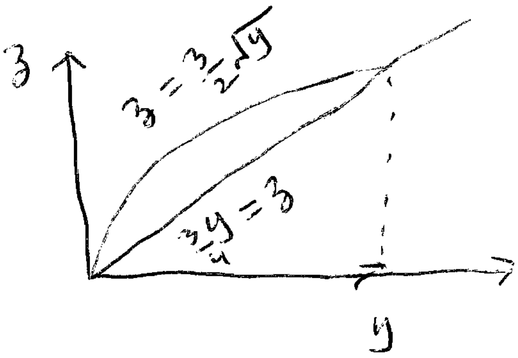
Q1) Determine the volume of the solid that lies below the plane $x+y+z=8$ and above the region in yz-plane that is bounded by $z=\frac{3}{2}\sqrt{y}$ and $z=\frac{3}{4}y$.

Sol. Here

$$0 \le y \le 4$$

$$\frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$$

$$0 \le x \le 8-y-z$$



Now $$V = \int_0^4 \int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} \int_0^{8-y-z} dx\,dz\,dy$$

$$= \int_0^4 \int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} (8-y-z)\,dz\,dy$$

$$= \int_0^4 \left(8z - yz - \frac{z^2}{2}\right)\Big|_{z=\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} dy$$

$$= \int_0^4 \left(12y^{\frac{1}{2}} - \frac{57}{8}y - \frac{3}{2}y^{\frac{3}{2}} + \frac{33}{32}y^2\right) dy$$

$$= \left[8y^{3/2} - \frac{57}{16}y^2 - \frac{3}{5}y^{\frac{5}{2}} + \frac{11}{32}y^3\right]_0^4 = \frac{49}{5}$$

Q2) Evaluate $\iiint y\,dV$ over the region E that lies below the plane $z = x+2$, above the xy-plane and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

<u>Sol.</u> Here $0 \le z \le x+2 \Rightarrow 0 \le z \le r\cos\theta + 2$

$$1 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le 2\pi$$

Therefore

$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r^2 \sin\theta \, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta\,(r\cos\theta+2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \left(\frac{1}{2}r^3\sin 2\theta + 2r^2\sin\theta\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{8}r^4\sin 2\theta + \frac{2}{3}r^3\sin\theta\right]_{r=1}^{2} d\theta$$

$$= \int_0^{2\pi}\left(\frac{15}{8}\sin 2\theta + \frac{14}{3}\sin\theta\right)d\theta$$

$$= \left[-\frac{15}{16}\cos 2\theta - \frac{14}{3}\cos\theta\right]_0^{2\pi} = 0$$

Q3. Evaluate $\iiint_E 16x\,dV$ where E is the upper half of the sphere $x^2+y^2+z^2=1$.

Sol. put $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Then $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{2}$. Therefore

$$I\ \iiint_E 16x\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1}(16\,\rho\sin\phi\cos\theta)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= 16\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$$

$$= 16\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[\frac{\rho^4}{4}\right]_{\rho=0}^{1}\sin^2\phi\cos\theta\,d\phi\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin^2\phi\cos\theta\,d\phi\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2\phi}{2}\right)\cos\theta\,d\phi\,d\theta$$

$$= 2\int_0^{2\pi}\left[\phi - \frac{\sin 2\phi}{2}\right]_{\phi=0}^{\frac{\pi}{2}}\cos\theta\,d\theta = \pi\int_0^{2\pi}\cos\theta\,d\theta$$

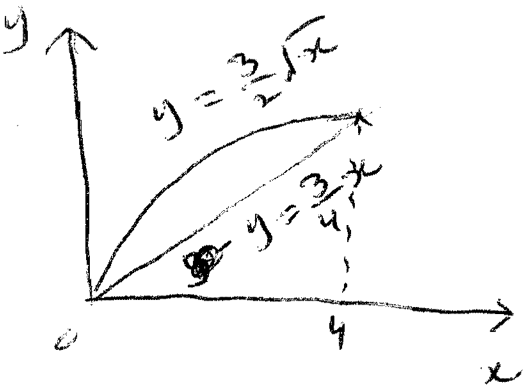
$$= \pi\left[\sin\theta\right]_0^{2\pi} = 0$$

Q1) Determine the volume of the solid that lies below the plane $x+y+z=8$ and above the region in xy-plane that is bounded by $y = \frac{3}{2}\sqrt{x}$ and $y = \frac{3}{4}x$.

<u>Sol:</u> Here $0 \le x \le 4$

$$\frac{3}{4}x \le y \le \frac{3}{2}\sqrt{x}$$

$$0 \le z \le 8 - x - y$$



Now
$$V = \int_0^4 \int_{\frac{3}{4}x}^{\frac{3}{2}\sqrt{x}} \int_0^{8-x-y} dz\, dy\, dx$$

$$= \int_0^4 \int_{\frac{3}{4}x}^{\frac{3}{2}\sqrt{x}} (8 - x - y)\, dy\, dx$$

$$= \int_0^4 \left[8y - xy - \frac{y^2}{2} \right]_{y=\frac{3}{4}x}^{\frac{3}{2}\sqrt{x}} dx$$

$$= \int_0^4 \left(12x^{\frac{1}{2}} - \frac{57}{8}x - \frac{3}{2}x^{\frac{3}{2}} + \frac{33}{32}x^2 \right) dx$$

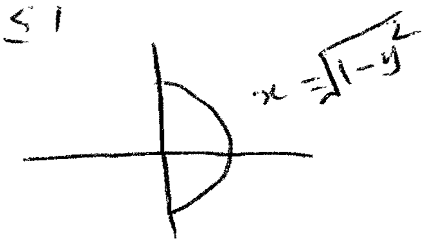
$$= \left[8x^{3/2} - \frac{57}{16}x^2 - \frac{3}{5}x^{\frac{5}{2}} + \frac{11}{32}x^3 \right]_0^4 = \frac{49}{5}$$

Q2) Evaluate $I = \iiint xyz \, dz\,dx\,dy$ where

$-1 \le y \le 1$, $0 \le x \le \sqrt{1-y^2}$, $x^2+y^2 \le z \le \sqrt{x^2+y^2}$.

<u>Sol</u>:- put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

Then $r^2 \le z \le r$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $0 \le r \le 1$



Therefore

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_{r^2}^{r} r^2 \sin\theta\cos\theta \, dz\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^2 \left[z\right]_{z=r^2}^{r} \sin\theta\cos\theta \, dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 (r^3 - r^4) \sin\theta\cos\theta \, dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4} - \frac{r^5}{5}\right]_{r=0}^{1} \sin\theta\cos\theta \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{4} - \frac{1}{5}\right) \sin\theta\cos\theta \, d\theta$$

$$= \frac{1}{20} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\cos\theta \, d\theta = \frac{1}{10} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2\theta \, d\theta = \frac{1}{10}\left[\frac{\cos 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 0$$

Q3) Evaluate $\iiint_E 16z\,dV$ where E is the upper half of the sphere $x^2+y^2+z^2=1$

Sol. put $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$. Then $0\le\rho\le1$, $0\le\theta\le2\pi$, $0\le\phi\le\frac{\pi}{2}$.

Therefore $\iiint_E 16z\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 16\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 8\rho^3\sin(2\phi)\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 2\left[\rho^4\right]_{\rho=0}^{1}\sin(2\phi)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 2\sin(2\phi)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\left[-\cos 2\phi\right]_{\phi=0}^{\frac{\pi}{2}}d\theta = -\int_0^{2\pi}(\cos\pi-\cos 0)\,d\theta$$

$$= -\int_0^{2\pi}(-2)\,d\theta$$

$$= 4\pi.$$