King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 690 - Final exam (121)   Time: 2 H 40 min

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Name: ID #

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Problem 1	/3
Problem 2	/7
Problem 3	/10
Problem 4	/10
Problem 5	/15
Total	/45

Problem # 1. (5 marks) Given two functions $u, v \subset H^1(\Omega)$, where Ω is a smooth and bounded domain of $\mathbb{R}^2$. Is uv in $H^1(\Omega)$? (Prove or disprove)

Problem # 2. (marks) Let $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. We would like to show that $f \in W^{1,p}(B)$, where $B = \{(x,y,z) \ / \ x^2+y^2+z^2 < 1\}$.

a. Find the largest q, for which $f \in L^q(B)$.

b. Find the largest p, for which $f \in W^{1,p}(B)$.

c. Any comments regarding the embedding results

Problem # 3. (10 marks) Let Ω be a bounded and smooth domain and let c_0 be the smallest constant satisfying

$$\int_\Omega |v|^2 = c_0 \int_\Omega |\nabla v|^2, \ \forall v \in H_0^1(\Omega).$$

Given f in $L^2(\Omega)$. Show that

$$(P) \begin{cases} -\Delta u + \lambda u = f, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases}$$

has a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$, provided that $\lambda > -1/c_0$

Problem # 4. (10 marks) Let Ω be a bounded and smooth domain and suppose that $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \ \forall v \in H^1(\Omega)$$

a. Use the regularity theorems to show that $u \in H^2(\Omega)$.

b. Show that

$$(*) \qquad \begin{cases} \Delta^2 u = f, & \text{in } \Omega \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$

Hint: You may put $w = \Delta u$ in (*)

c. Assume that Ω is C^4 and use the result of (b) and the regularity theorems to show that $u \in H^4(\Omega)$.

Problem # 5. (15 marks) Let Ω be a bounded and smooth domain, $f, g \in L^2(\Omega)$,. The aim is to prove the existence of a solution for the problem

$$(\mathrm{P}_1) \qquad \begin{cases} -\Delta u + v = f, & \text{in } \Omega \\ -\Delta v - u = g, & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & \text{on } \partial\Omega \end{cases}$$

a. Develop a concept of a weak solution (u, v) in an appropriate space to be determined and check that the Lax-Milgram theorem does not apply

b. To overcome this difficulty, we introduce the following approximating problem

$$(\mathrm{P}_2) \qquad \begin{cases} -\Delta u + v + \varepsilon u = f, & \text{in } \Omega \\ -\Delta v - u + \varepsilon v = g, & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & \text{on } \partial\Omega \end{cases}$$

Show that there exists a unique weak solution $(u^\varepsilon, v^\varepsilon) \in H^1(\Omega) \times H^1(\Omega)$ of problem (P_2).

c. Prove that

$$||u||_{L^2} + ||v||_{L^2} \leq ||f||_{L^2} + ||g||_{L^2}$$

d. Show that $(u^\varepsilon, v^\varepsilon)$ remains bounded in $H^2(\Omega) \times H^2(\Omega)$ as $\varepsilon \to 0$.

e. Show that (P_1) has a unique weak solution (u, v) in $H^2(\Omega) \times H^2(\Omega)$.

Theorem. Suppose that Ω is a domain of class C^2, with a bounded boundary. let $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega u\phi = \int_\Omega f\phi, \quad \forall \phi \in H^1(\Omega). \tag{0.1}$$

Then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2} \leq C\|f\|_{L^2},$$

where C is a constant depending on Ω only.

Moreover, if Ω is of class C^{m+2} and $f \in H^m(\Omega)$. Then

$$u \in H^{m+2}(\Omega), \qquad \text{and} \|u\|_{m+2} \leq C\|f\|_m.$$