King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 690 - Midterm Exam 1 (121)   Time: 2 H 20 min

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Name: ID #

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Problem 1	/5
Problem 2	/5
Problem 3	/5
Problem 4	/10
Problem 5	/10
Total	/35

Problem # 1. (5 marks) Given two sequences $(u_n) \subset L^p(\Omega)$ and $(v_n) \subset L^{p'}(\Omega)$, where $p > 1$ with $1/p + 1/p' = 1$ and Ω is a domain of $\mathbb{R}^N$. If $u_n \rightharpoonup u$ in $L^p(\Omega)$ and $v_n \to v$ in $L^{p'}(\Omega)$ show that $\int u_n v_n \to \int uv$

Problem # 2. (5 marks) Let $u(x) = |x| - 1$ and $v(x) = \frac{u(x)}{2+u(x)}$, $-1 < x < 1$. Show that $v \in W_0^{1,p}\left((-1,1)\right), \forall p \geq 1$.

Problem # 3. (5 marks) Let $I = (-1, 1)$. We define on $W_0^{1,1}(I)$, the linear functional F by $\langle F, v \rangle = v(0)$.

a. Show that F is well defined and bounded.

b. Find f_0, f_1 in $L^\infty(I)$ which satisfy

$$v(0) = \int_{-1}^{1} (f_0 v + f_1 v'), \ \forall v \in W_0^{1,1}(I).$$

Problem # 4. (10 marks) Let $I = (0, 1)$ and

$$V = \Big\{ v \in H^1(I) \ / \ v(1) = 0 \Big\}$$

a. Show that V is closed in $H^1(I)$
b. Show that $||v||_{L^2} \leq ||v'||_{L^2}$
c. If $f \in L^2(I)$ show that

$$(\mathrm{P}_1) \qquad \begin{cases} -u'' = f & \text{in } I \\ u'(0) = 0, & u(1) = 0 \end{cases}$$

has a unique solution $u \in V \cap H^2(\Omega)$

Problem # 5. (10 marks) Given the problem

$$(\mathrm{P}_2) \qquad \begin{cases} -u'' + v = f & \text{in } I = (0,1) \\ -v'' + v - u = g & \text{in } I \\ u(0) = 0, \qquad u(1) = 0 \\ v'(0) = 0, \qquad v'(1) = 0 \end{cases}$$

a. Show that, for $f, g \in L^2(I)$, problem (P_2) has a unique weak solution (u, v) in an appropriate space to be determined.

b. Show that if $f = g$ then $u - v \in H^4(I)$