King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 653 – Advanced Topics in Commutative Algebra (Term 122)

Exam 2 (Duration = 6 hours)

(Part 1 – 100/130)

Solve the following 5 problems.

(**1**) Prove that any integrally closed Noetherian domain of dimension ≤ 2 is Cohen-Macaulay.

(**2**) Let R be a Noetherian ring in which the classical height-unmixedness theorem holds (i.e., for any ideal I of height n which can be generated by n elements, all maximal primes belonging to I have height n and are minimal over I). Prove that R is Cohen-Macaulay.

(**3**) Let R be a regular local ring and let I be an ideal of R such that R/I is regular. Prove that $I = (x_1, \ldots, x_r)$ where the x's form part of a minimal generating set for the maximal ideal of R.

(**4**) A ring (not necessarily local) is Gorenstein if all its localizations with respect to maximal ideals are Gorenstein. Let R be a Noetherian ring (not necessarily local) and let x be an element of R such that $x \notin Z(R)$.

- **(a)** Assume R is local. Prove: R Gorenstein $\Leftrightarrow$ R/(x) Gorenstein.
- **(b)** Prove: R Gorenstein $\Rightarrow$ R/(x) Gorenstein.
- **(c)** Prove: $x \in J(R)$ and R/(x) Gorenstein $\Rightarrow$ R Gorenstein.

(**5**) Let R be a Noetherian ring (not necessarily local). The definition of a (global) Gorenstein ring is given in the above problem. Prove: $id_R(R) = n < \infty \Leftrightarrow$ R Gorenstein and dim(R) = n.

(Part 2 – 30/130)

Solve the following problem.

Let k be a field and let A and B be two k-algebras such that $A\otimes_k B$ is Noetherian. Let I and J be two proper ideals of A and B, respectively. Prove:

- **(a)** $G(I\otimes_k B) = G(I)$.
- **(b)** $G(I\otimes_k B + A\otimes_k J) = G(I) + G(J)$.
- **(c)** $G(I\otimes_k J) = \mathrm{Min}(G(I), G(J))$.

Homological basic facts that can be used for this exam (without proofs):

Fact 1: *If S is a multiplicatively closed subset of R and A, B are two R-modules with A finitely generated, then $S^{-1}Ext^n{}_R(A, B)$ is isomorphic to $Ext^n{}_{S^{-1}R}(S^{-1}A, S^{-1}B)$ for every positive integer n.*

Fact 2: *If A is an R-module, then: $id_R(A) \leq n \Leftrightarrow Ext^{n+1}{}_R(R/I, A) = 0$ for every ideal I of R.*