Department of Mathematics and Statistics

Math 653 – Advanced Topics in Commutative Algebra (Term 121)

Exam 1 (Duration = 6 hours)

(Part 1 – 70/100)

Solve 5 problems (out of 7) from the below list.

(**1**) Let $P = (p)$ be a principal prime ideal of a ring R and $J = \cap P^n$. Prove the following:

- **(a)** Q is a prime ideal of R strictly contained in $P \Rightarrow Q \subset J$.
- **(b)** $p \notin Z(R) \Rightarrow J = pJ$.
- **(c)** $p \notin Z(R) \Rightarrow J \in \mathrm{Spec}(R)$.
- **(d)** R is an integral domain and J is finitely generated $\Rightarrow J = 0$ and $\mathrm{ht}(P) = 1$.
- **(e)** J is finitely generated $\Rightarrow \mathrm{ht}(P) \leq 1$.

(**2**) Let R be a ring and $u \in U(R)$. Prove that $R[u] \cap R[u^{-1}]$ is integral over R.

(**3**) (The homogeneous Nullstellensatz) Let K be a field and $R = K[x_1, \ldots, x_n]$. An ideal I of R is homogeneous if : $f \in I \Rightarrow$ all the homogenous constituents of $f \in I$. A variety V ($\subseteq K^n$) is a cone if : $(a_1, \ldots, a_n) \in V \Rightarrow (ta_1, \ldots, ta_n) \in V$ for all $t \in K$.

- **(a)** Prove that if I is homogeneous, then $V(I)$ is a cone; where $V(I)$ denotes the variety consisting of all points of K^n where all polynomials in I vanish.
- **(b)** Prove that if V is a cone and K is infinite, then $J(V)$ is homogeneous; where $J(V)$ denotes the ideal of polynomials vanishing on V.
- **(c)** Prove that if K is algebraically closed, then the radical of a homogeneous ideal is homogeneous.

(**4**) Let R be an integral domain with quotient field K. Prove the following:

- **(a)** $S^{-1}R$ integral over $R \Rightarrow R = S^{-1}R$ (for any given multiplicative subset S of R).
- **(b)** Every ring between R and K is integrally closed $\Rightarrow R$ is Prüfer.
- **(c)** Every ring between R and K is a localization $\Rightarrow R$ is Prüfer.

(**5**) Let R be an integral domain, K its quotient field, and R' its integral closure. Let T be a ring between R and K, and D the conductor of T relative to R, i.e., $D = (R :_R T)$. Prove:

- **(a)** R is Noetherian and $D \neq 0 \Rightarrow T$ is a finitely generated R-module
- **(b)** $D \not\subset P \in \mathrm{Spec}(R)$ and $Q \in \mathrm{Spec}(T)$ with $Q \cap R = P \Rightarrow R_P = T_Q$.
- **(c)** T is a finitely generated R-module and $P \in \mathrm{Spec}(R)$ with $R_P = T_P \Rightarrow D \not\subset P$.
- **(d)** If R' is a f.g. R-module and $P \in \mathrm{Spec}(R)$, then: R_P is integrally closed $\Leftrightarrow (R :_R R') \not\subset P$.

(**6**) Let R be an integral domain with quotient field K. Suppose that every ring between R and K is Noetherian. Prove that the Krull dimension of R is at most 1. (This is the converse of Theorem 93.)

(**7**) Let an integral domain $R = \cap R_i$ be a locally finite intersection of one-dimensional local domains lying between R and its quotient field. For each i, let M_i denote the maximal ideal of R_i and let $P_i = M_i \cap R$.

(a) Prove that any non-zero element of R lies in only a finite number of minimal prime ideals of R.

(b) Assume $ht(P_i) = 1$ for each i. Prove that for any $0 \neq a \in R$, we have

$$Z(R/(a)) = P_1 \cup \dots \cup P_n$$

where $P_1, \dots, P_n$ are the minimal prime ideals containing a.

(c) Prove that every prime ideal of grade 1 has height 1.

(Part 2 – 30/100)

Solve the following problem.

Let D be an integral domain with quotient field K and E a subset of K.

- Let $Int(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$ called the ring of D-integer-valued polynomials over E. If E = D, we write Int(D) instead of Int(D, D).
- The polynomial closure of E in D is the largest subset F of K with Int(E, D) = Int(F, D) and is given by $cl_D(E) = \{x \in K \mid f(x) \in D \text{ for each } f \in Int(E, D)\}$.
- E is said to be D-fractional if $\exists\, 0 \neq d \in D$ such that $dE \subseteq D$. So $dX \in Int(E, D)$.
- E is polynomially dense in D if Int(E, D) = Int(D). (e.g., N is polynomially dense in Z.)

Assume that for some $\Delta \subseteq Spec(D)$, $D = \bigcap_{p\in\Delta} D_p$ is a locally finite representation of D. Then prove the following:

(a) $Int(E, D)_p = Int(E, D_p)$, for each $p \in \Delta$.

(b) $cl_D(E) = \bigcap_{p\in\Delta} cl_{D_p}(E_p)$.

(c) If $E \subseteq D$, then: E is polynomially dense in D $\Leftrightarrow$ E is polynomially dense in D_p $\forall\, p \in \Delta$.
