KFUPM - Department of Mathematics and Statistics
Final Exam MATH 552, Fields and Galois Theory, Term 121
Duration: 180 minutes

NAME:
ID:

Solve the following Exercises.

Exercise 1(15 points). Let $K|_F$ be an extension of fields. Prove that K is algebraic over F if and only if every intermediate ring between F and K is a field.

Exercise 2 (20 points). Let $K|_F$ be a quadratic extension of fields and suppose that F is of characteristic 2.
(1) Prove that K is separable over F if and only if there is $\theta \in K \setminus F$ and $\alpha \in F$ such that $\theta^2 + \theta + \alpha = 0$.
(2) Prove that $K = F(\theta)$ and $K|_F$ is Galois.
(3) Prove that $X^2 + X + \alpha = (X + \theta)(X + \theta + 1)$ and it is the minimal polynomial of θ over F.

Exercise 3(20 points).
(1) Prove that for any root θ of the polynomial $X^4 - 2$ of $\mathbb{Q}[X]$, $\mathbb{Q}(\theta)$ is not a normal extension of $\mathbb{Q}$.
(2) Find three extensions $\mathbb{K}_1 \subsetneqq \mathbb{K}_2 \subsetneqq \mathbb{K}_3$ such that $\mathbb{K}_2$ is normal over $\mathbb{K}_1$, $\mathbb{K}_1$ is normal over $\mathbb{Q}$ but $\mathbb{K}_2$ is not normal over $\mathbb{Q}$.
(3) $\mathbb{K}_3$ normal over $\mathbb{Q}$.

Exercise 4(25 points). Recall that if $\mathbb{E}$ is a subset of the complex field $\mathbb{C}$ containing 0 and 1 and $\hat{\mathbb{E}}$ is the set of all element $z \in \mathbb{C}$ constructible (from $\mathbb{E}$) with ruler and compass, then $\mathbb{E}$ is the smallest subfield of $\mathbb{C}$ containing $\mathbb{E}$ and satisfying:
(i) if $z^2 \in \hat{\mathbb{E}}$, then $z \in \hat{\mathbb{E}}$, (ii) if $z \in \hat{\mathbb{E}}$, then $\bar{z} \in \hat{\mathbb{E}}$.
(1) Prove that an angle θ is constructible (from $\mathbb{E}$) if and only if $cos\theta$ is constructible.
(2) Find a necessary and sufficient condition so that the angle θ is constructible from the angle 3θ.
(3) Is the angle $\theta = 10$ constructible from the angle 30?

Exercise 5(20 points). Let $f(X) = X^3 - 3X + 1 \in \mathbb{Q}[X]$, and let K be the splitting field of $f(X)$ over $\mathbb{Q}$.
(1) Prove that $f(X)$ is solvable by radicals.
(2) Prove that K is not a radical extension of $\mathbb{Q}$.

Exercise 6(20 points) Let F be a field containing a primitive nth root of unity and K a finite extension of F. Prove that $K|_F$ is an n-Kummer extension if and only if $K = F(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_r})$ for some $a_i \in F$.

Exercise 7(30 points). Let F be a field, N an algebraic closure of F and $a \in N$ separable over F. Let K be the normal closure of $F(a)$ over F, p a prime positive integer, $H_1, \ldots, H_s$ the p-Sylow subgroups of $Gal(K|_F)$, H the subgroup of $Gal(K|_F)$ generated by $\bigcup_{i=1}^{i=s} H_i$, $M_i = \mathcal{F}(Hi)$ (the fixed field of Hi) and $M = \mathcal{F}(H)$.

(1) Prove that $M = \bigcap_{i=1}^{i=s} M_i$ and that M is Galois over F.

(2) Prove that if p divides $[K : F]$, then there exists $j \in \{1, \ldots, s\}$ such that $a \notin M_j$.

(3) Let L be an intermediate field of $N|_F$.

(i) Prove that $K \cap L(a) = (K \cap L)(a)$.

(ii) Prove that if $L(a)|_L$ is Galois, then $(K \cap L)(a)|_{K \cap L)}$ is Galois and $Gal(K|_L)$ is isomorphic to a subgroup of $Gal((K \cap L)(a)|_{K \cap L})$.

(4) Prove that p divides $[K : F]$ if and only if there is an intermediate field L of $N|_F$ such that $L(a)|_L$ is Galois of degree p.
