KFUPM - Department of Mathematics and Statistics
MATH 552, Term 121
Exam II (Out of 50)

NAME:
ID:

Solve the following Exercises.

Exercise 1 (10 points 5-5): Let K be an extension of F, and $[F, K]$ the set of all intermediate fields between F and K.
(1) Prove that if $K = F(a, b)$ for some $a, b \in K$ and $[F, K]$ is finite, then K is a simple extension of F.
(2) Prove that if $[F, K]$ is finite, then K is a simple extension of F. (Hint, use Question 1).

Exercise 2 (10 points 5-5): Let F be a finite field of order q and let $f \in F[X]$ an irreducible polynomial over F.
1-Prove that, for any positive integer n, f divides $X^{q^n} - X$ if and only if the degree of f divides n.
2-Prove that, for any positive integer n, $F[X]$ contains an irreducible polynomial of degree n.

Exercise 3 (10 points 5-5): Let K be a finite field of order 4.
(1) Prove that $K = \mathbb{F}_2(\alpha)$ where $\alpha^2 + \alpha + 1 = 0$.
(2) Find the irreducible factorization of $X^4 + 1$ over $\mathbb{F}_3$.

Exercise 4 (10 points 3-3-4): (1) Let K be a purely inseparable extension of F. Prove that $Tr(a) = 0$ for any $a \in K$.
(2) Let $K|_F$ be an extension of finite fields. Prove that the norm map $N_{K|F}$ is surjective.
(3) Let p be an odd prime, ω a primitive pth root of unity and $K = \mathbb{Q}(\omega)$. Prove that $Nr_{K|_\mathbb{Q}}(1 - \omega) = p$.

Exercise 5 (10 points 3-3-4): Let F be a field.
(1) Prove that if $a \in F$, then the splitting field of $X^n - a$ over F is $F(b, \omega)$ where $b^n = a$ and ω is a primitive nth root of unity.
(2) Set $N = F(b, \omega)$, $K = F(b)$ and $L = F(\omega)$. Prove that the extension $L|_F$ is Galois and the extension $N|_L$ is cyclic.
(3) Suppose that the minimal polynomial of ω over F is $P_{F,\omega}(X) = (X-\omega)(X-\omega^{-1})$ and that $[N : L] = n$ Prove that there exits $\sigma \in Gal(N|_F)$ of order n such that $\sigma(b) = b\omega$ and $\sigma(\omega) = \omega$.