KFUPM - Department of Mathematics and Statistics
MATH 552, Term 121
Exam I (Out of 100)

NAME:
ID:

Solve the following Exercises.

Exercise 1 (10 points 3-3-4): Let $K = F(a)$ be a finite extension of F. For $\alpha \in K$, define $\phi_\alpha : K \to K$ by $\phi_\alpha(x) = x\alpha$.
(1) Prove that ϕ_α is an F-linear transformation.
(2) Prove that $det(xI - \phi_a)$ is the minimal polynomial of a, where I is the identity F-transformation.
(3) For which $\alpha \in K$ is $det(xI - \phi_\alpha)$ the minimal polynomial of α over F.

Exercise 2 (10 points): Let $K|_F$ be a field extension.
Prove that K is algebraic over F if and only if every monic polynomial of $K[X]$ is a divisor of some non-zero polynomial in $F[X]$.

Exercise 3 (10 points 3-4-3): Let a be an odd integer.
(1) Prove that $X^3 + 4X - a^2$ is irreducible in $\mathbb{Q}[X]$.
(2) Let $\mathbb{E}$ be a splitting field of $X^4 - aX - 1$, θ a root of $X^4 - aX - 1$ in $\mathbb{E}$ and $K = \mathbb{Q}(\theta)$. Prove that for every α, β in $\mathbb{E}$, if $X^2 + \alpha + \beta$ divides $X^4 - aX - 1$, then the minimal polynomial $P_{\alpha^2,\mathbb{Q}}$ of α^2 over $\mathbb{Q}$ is $X^3 + 4X - a^2$.
(3) Find $[K : \mathbb{Q}]$.

Exercise 4 (10 points 3-3-4): Let F be a field of positive characteristic p, $K = F(X, Y)$ and $L = F(X^p, Y^p)$.
(1) Find $[K : L]$.
(2) Prove that $u^p \in L$ for every $u \in K$.
(3) Prove that the extension $K|_L$ has infinitely many intermediate fields.

Exercise 5 (10 points): Let K be a field and $\sigma \in Aut(K)$ has infinite order. Let F be the fixed field of σ. Prove that if K is algebraic over F, then K is a normal extension of F.

Exercise 6 (10 points 5-5): Let K and L be two normal extensions of a field F and suppose that K and L are subfields of a common field $\mathbb{E}$.
(1) Prove that $K \cap L$ is a normal extension of F.
(2) Prove that $F(K \cup L)$ is a normal extension of F.

Exercise 7 (10 points 3-4-3):
(1) Prove that for any root θ of the polynomial X^4-2 of $\mathbb{Q}[X]$, $\mathbb{Q}(\theta)$ is not a normal extension of $\mathbb{Q}$.
(2) Find three extensions $\mathbb{K}_1 \subsetneqq \mathbb{K}_2 \subsetneqq \mathbb{K}_3$ such that $\mathbb{K}_2$ is normal over $\mathbb{K}_1$, $\mathbb{K}_1$ is normal over $\mathbb{Q}$ but $\mathbb{K}_2$ is not normal over $\mathbb{Q}$.
(3) $\mathbb{K}_3$ normal over $\mathbb{Q}$.

Exercise 8 (10 points 3-4-3): Let $K|_F$ be an extension of fields of characteristic $p \neq 0$ and set $L = \{x \in K | x^{p^r} \in F$ for some integer $r \geq 0\}$.
(1) Prove that L is an intermediate field (i. e., $F \subseteq L \subseteq K$.
(2) Prove that if K is perfect, then L is perfect.
(3) Prove that every F-automorphism of K is an L-automorphism.

Exercise 9 (10 points): Let $F \subseteq L \subseteq K$ be extension of fields such that $K|_L$ is normal and $L|_F$ is purely inseparable. Prove that $K|_F$ is normal.

Exercise 10 (10 points 3-4-3): Let F be a field of characteristic $p \neq 0$, n a positive integer and $a \in F$.
(1) Prove that X^{p^n} has only one irreducible divisor f in $F[X]$.
(2) Prove that there exists a positive integer $m \leq n$ such that $X^{p^n} - a = f^{p^m}$.
(3) Prove that there exists $b \in F$ such that $f = X^{p^{n-m}} - b$.