King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH533 - Complex Variables
HomeWork 2 – Semester I, 2012-2013

Exercise 1

Find all entire functions f such that

$$|f(z)| \leq \frac{|z|^4}{\ln|z|} \quad \text{for} \quad |z| > 1.$$

Exercise 2

Prove that for any polynomial $p(z)$

$$\frac{1}{2i\pi}\int_{\partial B(c)} \overline{p(z)}\, dz = \overline{p'(c)}$$

where $B(c) = \{z \in \mathbb{C} : |z - c| < 1\}$, $c \in \mathbb{C}$.

Exercise 3

Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be $\gamma(t) = r\, e^{it}$ and let $g : |\gamma| \to \mathbb{C}$ be continuous. Show that

$$\overline{\int_\gamma g(w)\, dw} = -\int_\gamma \overline{g(w)} \left(\frac{r^2}{w^2}\right) dw.$$

Exercise 4

Let f be analytic on the closure of the disc $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$. Define

$$g(z) = \frac{1}{2\pi i}\int_{\partial\Delta_r} \frac{\overline{f(w)}}{w - z}\, dw, \quad |z| \neq r.$$

1. Show that
$$\overline{g(z)} = \frac{1}{2\pi i}\int_{\partial\Delta_r} \left(\frac{f(w)}{\overline{w} - \overline{z}}\right) \frac{r^2}{w^2}\, dw.$$
2. Find $g(z)$ for $|z| > r$ and prove that $g(z) = \overline{f(0)}$ for $|z| < r$.
3. Deduce that
$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta_r} \frac{\Re f(w)}{w} \frac{(w+z)}{w-z}\, dw + i\Im f(0), \text{ for } |z| < r.$$
(Hint: $\Re f = \dfrac{f + \bar{f}}{2}$)

Exercise 5

Suppose that f is an entire function such that $f(0) = 1$, $f'(0) = 0$ and $f''(1 + \frac{1}{n}) = 7 - \frac{3}{n}$ for all n natural number. Find f that satisfies these properties.

Exercise 6

Let f be analytic in some region containing the closed unit disc $\overline{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$ and suppose that $|f(z)| \leq M$ for all $z \in \Delta$.

1. If $f(a_k) = 0$ for $1 \leq k \leq n$, show that

$$|f(z)| \leq M \prod_{k=1}^{n} \left| \frac{z - a_k}{1 - \bar{a}_k z} \right|.$$

2. If $f(a_k) = 0$ for $1 \leq k \leq n$, each $a_k \neq 0$ and $|f(0)| = M|a_1 \ldots a_n|$, find a formula for f.

Exercise 7

Suppose that both f and g are analytic on $\overline{\Delta_r} = \{z \in \mathbb{C} : |z| \leq r\}$, with $|f(z)| = |g(z)|$ for $|z| = r$. Show that if neither f nor g vanishes in Δ_r, then there is a constant λ, $|\lambda| = 1$, such that $f = \lambda g$.

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