KFUPM/ Department of Mathematics and Statistics ID#
MATH 460 / Final Test/ 121 Name:

Q1. (8 points) (a) $Ax = b$ has a solution under what conditions on b, if $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$?

(b) Find a basis for the null space of A.

(c) Find the general solution to Ax = b, when a solution exists.

(d) Find a basis for the column space of A.

(e) What is the rank of A^T ?

Q2. (6 points) Apply the Gram-Schmidt process to $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and write the result as A=QR.

Q3. (4 points) Find the least square solution of Ax=b if A=$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

Q4. (6 points) If A has eigenvalues 0 and 1, corresponding to the eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ how can you tell in advance that A is symmetric? What are its trace and determinant? What is A?

Q5. (5 points) Prove that every unitary matrix A is diagonalizable in two steps:

(i) If A is unitary, and U is too, then so is $T = U^{-1}AU$

(ii) An upper triangular T that is unitary must be diagonal.

Q6. (5 points) Prove that a matrix with orthonormal eigenvectors has to be normal.

Q7. (5 points) Compute J^{10} and A^{10} if $A = MJM^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} =$

$$\begin{bmatrix} 14 & 9 \\ -16 & -10 \end{bmatrix}$$

Q8. (10 points) Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

a) Is A symmetric?
b) Show that A is positive definite by three ways
c) Find the symmetric Factorization $A = LDL^T$ of A
d) Using c) Find R such that $A = R^T R$

Q9. (11 points) Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

a) Find $\|A\|_2$, $\|A\|_1$ and $\|A\|_\infty$

b) Find the condition number in terms of the $\|A\|_2$

c) Consider the system $Ax = b$ where $b = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ and suppose we pertub b so that it changes to $b' = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

(i) Without solving $Ax = b$ or $Ax = b'$, give an upper bound on the relative error $\dfrac{\|\delta x\|}{\|x\|}$.

(ii) Solve the system $Ax = b$ and $Ax = b'$ to determine the exact relative error.

Q10. (7 points) Let $H = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and let $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 1$ be the eigenvalues of H.

a) Use Rayleigh Quotient to confirm λ_1 and λ_3
b) Use cournat Minimax theorem to confirm λ_2

Q11. (4 points) If $H = \begin{bmatrix} 1.43 & -7.89 \\ -7.89 & 8.36 \end{bmatrix}$ and $M = \begin{bmatrix} 1.4 & -7.9 \\ -7.9 & 8.4 \end{bmatrix}$, then use Weyl's theorem to find $|\mu_i - \lambda_i|$ where μ_i and λ_i are the eigenvalues of H and M respectively.

Q12. (5 points) Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

a) State Gershgorin's Theorem.
b) Use it to show that zero cannot lie in any of the Gershgorin circles.
c) Is A singular or nonsingular? Why?

Q13. (15 points) Let $A = \begin{bmatrix} \frac{-1}{3} & \frac{4000}{3} \\ \frac{2}{3000} & \frac{1}{3} \end{bmatrix}$ and $\delta A = \begin{bmatrix} \frac{-1}{1000} & \frac{-1}{1000} \\ \frac{2}{1000} & \frac{-1}{1000} \end{bmatrix}$. Compute the variations $\delta\lambda_1$, $\delta\lambda_2$, $\delta\mu_1$ and $\delta\mu_2$.

Q14. (9 points) Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

a) Find the Jacobi Matrix and its eigenvalues
b) Find the Gauss-Seidel matrix and its eigenvalues
c) Find ω_{opt} and $\lambda_{\max}$ for SOR.

Q15. (10 points)

Let $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

a) Find the singular value decomposition of A, giving that $Q_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$.

b) Find the pseudoinverse of A.

c) Show that if M is an m x n and Q is an orthogonal m x m matrix then M and QM have the same singular values.