King Fahd Univ. of Petroleum and Minerals
Faculty of Sciences
Department of Mathematics and Statistics

FINAL EXAM
MATH. 411-121

**Name:**

**ID:**

**Prob. 1**

Let $m$ be a positive integer, $f(x) = e^x$ and $Y = I_1 \cup I_2 \cup ... \cup I_m$ where $I_k = [(k-1)/m, k/m] \times [0, f(k/m)]$, $k = 1, ..., m$. Find the area $V_2(Y)$. Show that it is approximately $e - 1$ if $m$ is large.

**Prob. 2**

Show that if $A$ is measurable and $B$ is a null set, then $V(A \cup B) = V(A \backslash B) = V(A)$.

**Prob. 3**

Let $[a]$ denote the largest integer which is not greater than $a$. Let $\Phi(x, y) = [x + y]$ if $0 \leq x < r$, $0 \leq y < s$ where $r$ and $s$ are positive integers. For all other $(x, y)$ let $\Phi(x, y) = 0$. Show that

$$\int \Phi dV_2 = \frac{rs(r + s - 1)}{2}.$$

**Prob. 4**

Show that $f$ is integrable over $A$

(a) $f(x) = \sin \frac{1}{x}$ if $x \neq 0$, $f(0) = 5$, $A = [-1, 1]$

(b) $f(x, y) = \frac{x^4 - y^2}{x^2 - y}$, $A = \{(x, y) : |x| \leq 1, |y| \leq 1, x^2 \neq y\}$

**Prob. 5**

Find the volume of $\{(x, y, z) : |x| + |y| + |z| \leq 2, z^2 \leq y\}$.

**Prob. 6**

Let $f_n(x) = n/(x^2+n^2)$, $n = 1, 2, ...$ Show that $0 \leq f_n(x) \leq 1$, $\lim_{n\to\infty} f_n(x) = 0$, for all $x$ and $\int f_n dx = \pi$. Why does this not contradict the DCT?

**Prob. 7**

Let $f_1, f_2, ...$ be functions on $R$ which satisfy the hypotheses of Lebesgue's dominated convergence theorem. Let $F_n(x) = \int_0^x f_n(t)dt$, $F(x) = \int_0^x f(t)dt$. Show that $F_n$ tends to $F$ uniformly on $R$ as $n \to \infty$.

**Prob. 8**

For $x \in R$, let $\Phi(x) = \int_0^\infty e^{-t^2} \cos(xt)dt$. Show that $\Phi'(x) = -\frac{1}{2}x\Phi(x)$ and find $\Phi(x)$.

**Prob. 9**

Let $\Phi(t) = \int_0^1 \log(2-x^2t^2)dx$. Show that $\Phi'(0) = 0$ and that $\Phi$ is concave on $(-1, 1)$.

**Prob. 10**

Let $f(x) = x^{-1}(\log x)^{-2}$, $A = (0, \frac{1}{2})$. Show that $f \in L^p(A)$ for $p = 1$ but not for $p > 1$.