MATH 301.2 (Term 121)

Quiz 5 (Sects. 12.5-12.6) Duration: 20mn

Name: **ID number:**

1.)(5pts)Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(1) = 0$.

2.)(5pts)Expand $f(x) = x$, $0 < x < 1$, **in a Fourier Bessel series using Bessel functions of order one satisfying the boundary conditions** $J_1(\alpha) + \frac{\alpha}{2}J_1'(\alpha) = 0$

3.)(5pts)Write the first four terms of the Fourier-Legendre expansion of

$$f(x) = \begin{cases} x, & -1 \le x < 0, \\ 0, & 0 < x \le 1. \end{cases}$$

1.) $m^2 + \lambda = 0$

- if $\lambda = 0$, $m^2 = 0$, $m = 0$

$y = c_1 + c_2 x$

$y(0)+y'(0) = 0 \Rightarrow c_1 + c_2 = 0$, $y(1) = 0 \Rightarrow c_1 + c_2 = 0$ $\Big\} \Rightarrow c_2 = -c_1$

$y = c_1(1-x)$

$\lambda = 0$ is an eigenvalue and $y_0 = 1-x$ is the corresponding eigenfunction

- if $\lambda < 0$, $\lambda = -\alpha^2$, $m^2 = \alpha^2$

$m = \pm\alpha$

$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$y(0)+y'(0) = 0 \Rightarrow c_1 + c_2 + \alpha(c_1 - c_2) = 0$

$y(1) = 0 \Rightarrow c_1 e^{\alpha} + c_2 e^{-\alpha} = 0$

$\Rightarrow c_1 = c_2 = 0$

and $y = 0$

- If $\lambda > 0$, $\lambda = \alpha^2$, $m = \pm i\alpha$

$y = c_1 \cos\alpha x + c_2 \sin\alpha x$

$y(0) + y'(0) = 0 \Rightarrow c_1 + \alpha c_2 = 0$

$y(1) = 0 \Rightarrow c_1 \cos\alpha + c_2 \sin\alpha = 0$

$c_1 = -\alpha c_2$, $-\alpha c_2 \cos\alpha + c_2 \sin\alpha = 0$

$c_2 \neq 0$ $\tan\alpha = \alpha$

There exist α_n / $\tan\alpha_n = \alpha_n$

$\lambda_n = \alpha_n^2$ are the eigenvalues

and $y_n = -\alpha_n \cos\alpha_n x + \sin\alpha_n x$ are the corresponding eigenfunctions.

2.) $J_1(\alpha) + \frac{\alpha}{2} J_1'(\alpha) = 0 \Leftrightarrow 2J_1(\alpha) + \alpha J_1'(\alpha) = 0$

$n = 1$, $b = 1$, $h = 2$

$f(x) = \sum_{i=1}^{\infty} c_i J_1(\alpha_i x)$,

$c_i = \dfrac{2\alpha_i^2}{(\alpha_i^2 - 1 + 4) J_1^2(\alpha_i)} \int_0^1 x J_1(\alpha_i x) f(x)\, dx$

$= \dfrac{2}{\alpha_i(\alpha_i^2 + 3) J_1^2(\alpha_i)} \int_0^{\alpha_i} u^2 J_1(u)\, du$

but, $\int_0^{\alpha_i} u^2 J_1(u)\, du = \alpha_i^2 J_2(\alpha_i)$

$C_i = \dfrac{2\alpha_i \, J_2(\alpha_i)}{(\alpha_i^2 + 3) J_1^2(\alpha_i)}$

3.) $f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$, $\quad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) P_n(x)\,dx$

$$c_0 = \frac{1}{2}\int_{-1}^{1} f(x) P_0(x)\,dx = \frac{1}{2}\int_{-1}^{0} x\,dx = -\frac{1}{4}$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} f(x) P_1(x)\,dx = \frac{3}{2}\int_{-1}^{0} x^2\,dx = \frac{1}{2}$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} f(x) P_2(x)\,dx = \frac{5}{2}\int_{-1}^{0} \frac{x}{2}(3x^2-1)\,dx = -\frac{5}{16}$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} f(x) P_3(x)\,dx = \frac{7}{2}\int_{-1}^{0} \frac{x}{2}(5x^3-3x)\,dx = 0$$

MATH 301.3 (Term 121)

Quiz 5 (Sects. 12.5-12.6) Duration: 20mn

Name: **ID number:**

1.)(5pts)Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(1) = 0$.

2.)(5pts)Expand $f(x) = x$, $0 < x < 3$, **in a Fourier Bessel series using Bessel functions of order one satisfying the boundary conditions** $J_1(3\alpha) + \alpha J_1'(3\alpha) = 0$

3.)(5pts)Write the first four terms of the Fourier-Legendre expansion of

$$f(x) = \begin{cases} 0, & -1 \le x < 0, \\ -x, & 0 < x \le 1. \end{cases}$$

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1) See Quiz 5, section 2

2) $J_1(3\alpha) + \alpha J_1'(3\alpha) = 0$

$\Leftrightarrow 3J_1(3\alpha) + 3\alpha J_1'(3\alpha) = 0$

$b = 3, \quad h = 3$

$f(x) = \sum_{i=1}^{\infty} c_i J_1(\alpha_i x)$,

$c_i = \dfrac{2\alpha_i^2}{(9\alpha_i^2 - 1 + 9) J_1^2(3\alpha_i)} \int_0^3 x J_1(\alpha_i x)\, f(x)\, dx$

$= \dfrac{2\alpha_i^2}{(9\alpha_i^2 + 8) J_1^2(\alpha_i 3)} \int_0^3 x^2 J_1(\alpha_i x)\, dx$

$= \dfrac{2}{\alpha_i (9\alpha_i^2 + 8) J_1^2(3\alpha_i)} \int_0^{3\alpha_i} u^2 J_1(u)\, du$

But, $\dfrac{d}{dx}\left[x^2 J_2(x)\right] = x^2 J_1(x)$

$\Rightarrow \int_0^{3\alpha_i} u^2 J_1(u)\, du = 9\alpha_i^2 J_2(3\alpha_i)$

$\Rightarrow c_i = \dfrac{18\alpha_i \quad J_2(3\alpha_i)}{(9\alpha_i^2 + 8) J_1^2(3\alpha_i)}$

3) $f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$, $c_n = \dfrac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$

$c_0 = \dfrac{1}{2} \int_{-1}^{1} f(x) P_0(x)\, dx = \dfrac{1}{2} \int_0^1 (-x)\, dx = -\dfrac{1}{4}$

$c_1 = \dfrac{3}{2} \int_{-1}^{1} f(x) P_1(x)\, dx = \dfrac{3}{2} \int_0^1 (-x^2)\, dx = -\dfrac{1}{2}$

$c_2 = \dfrac{5}{2} \int_{-1}^{1} f(x) P_2(x)\, dx = \dfrac{5}{2} \int_0^1 \dfrac{-x}{2}(3x^2 - 1)\, dx = -\dfrac{5}{16}$

$c_3 = \dfrac{7}{2} \int_{-1}^{1} f(x) P_3(x)\, dx = \dfrac{7}{2} \int_0^1 \dfrac{-x}{2}(5x^3 - 3x)\, dx = 0$