King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Major Exam 2
The First Semester of 2012-2013 (121)

Time Allowed: 120 Minutes

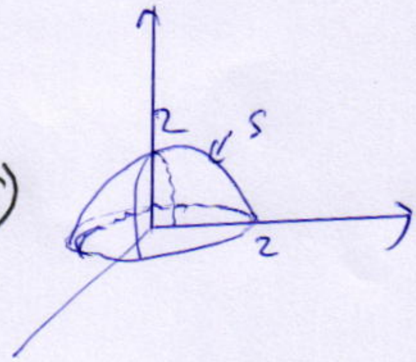
Name: ______________________ ID#: ______________

Instructor: ______________________ Sec #: ________ Serial #: ________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

| Question # | Marks | Maximum Marks |
|---|---|---|
| 1 | | 12 |
| 2 | | 16 |
| 3 | | 16 |
| 4 | | 12 |
| 5 | | 14 |
| 6 | | 16 |
| 7 | | 14 |
| Total | | 100 |

**Q:1** (12 points) Let $\overrightarrow{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ be a vector field and $S$ the surface of the region bounded by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the plane $z = 0$. Use divergence theorem to evaluate $\int\int_S (\overrightarrow{F} \cdot \hat{n})dS$.




The Divergence theorem says

$$\iint_S F.n\,dS = \iiint_D \operatorname{div} F\, dV \quad (3\text{ pts})$$

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$= 3x^2 + 3y^2 + 3z^2 \quad (3\text{ pts})$$

$$\iint_S F.n\,dS = 3\iiint_D (x^2+y^2+z^2)\, dV$$

We may use spherical coordinates. This gives

$$\iint_S F.n\,dS = 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \quad (2\text{ pts})$$

$$= 3\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left[\frac{\rho^5}{5}\right]_0^2 d\phi\, d\theta$$

$$= 3\cdot\frac{32}{5}\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2} d\theta \quad (2\text{ pts})$$

$$= \frac{192\pi}{5} \quad (2\text{ pts})$$

**Q:2** (4+6+6 points) Find the following:

**(a)** $\mathcal{L}\{e^{-t}\cosh t\}$,

**(b)** $\mathcal{L}^{-1}\left\{\dfrac{s-1}{s^2+2s}\right\}$,

**(c)** $\mathcal{L}^{-1}\left\{\dfrac{s+1}{(s-1)(s^2+1)}\right\}$.

a) $\mathcal{L}\{e^{-t}\cosh t\} = \dfrac{s+1}{(s+1)^2-1} = \dfrac{s+1}{s(s+2)}$ 4pts

b) $\dfrac{s-1}{s^2+2s} = \dfrac{s-1}{s(s+2)} = \dfrac{-1/2}{s} + \dfrac{\frac{3}{2}}{s+2}$ 3pts

$\mathcal{L}^{-1}\left\{\dfrac{s-1}{s^2+2s}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$= -\frac{1}{2} + \frac{3}{2}e^{-2t}$ 3pts

c) $\dfrac{s+1}{(s-1)(s^2+1)} = \dfrac{1}{s-1} - \dfrac{s}{s^2+1}$ 3pts

$\mathcal{L}^{-1}\left\{\dfrac{s+1}{(s-1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\}$

$= e^{t} - \cos t$ 3pts

**Q:3** (4+5+7 points) Find the following:

**(a)** $\mathcal{L}\{te^{-t}\cos t\}$,

**(b)** $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2, & t \ge 1 \end{cases}$.

**(c)** $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+1)^2}\right\}$.

a) $\mathcal{L}\{te^{-t}\cos t\} = -\dfrac{d}{ds}\mathcal{L}\{e^{-t}\cos t\} = -\dfrac{d}{ds}\left[\dfrac{s+1}{(s+1)^2+1}\right] = \dfrac{s^2+2s}{(s^2+2s+2)^2}$ 2pts 2pts 4pts

b) $f(t) = t - t\mathcal{U}(t-1) + 2\,\mathcal{U}(t-1)$ 2pts

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{t\mathcal{U}(t-1)\} + \mathcal{L}\{2\,\mathcal{U}(t-1)\}$

$= \dfrac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\} + e^{-s}\mathcal{L}\{2\}$

$= \dfrac{1}{s^2} - \left(\dfrac{1}{s^2}+\dfrac{1}{s}\right)e^{-s} + \dfrac{2}{s}e^{-s}$

$= \dfrac{1}{s^2} - \left(\dfrac{1}{s^2}-\dfrac{1}{s}\right)e^{-s}$ 3pts

c) $\dfrac{s^2}{(s^2+1)^2} = \dfrac{s}{s^2+1}\cdot\dfrac{s}{s^2+1}$

$\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+1)^2}\right\} = \cos t * \cos t$ 2pts

$= \int_0^t \cos\tau\,\cos(t-\tau)\,d\tau$ 1pt

$$\left.\begin{aligned}
&= \frac{1}{2}\int_0^t [\cos t + \cos(2\tau - t)]\,d\tau \\
&= \frac{1}{2}\left(t\cos t + \left[\frac{\sin(2\tau-t)}{2}\right]_0^t\right) \\
&= \frac{1}{2}\left(t\cos t + \frac{\sin t}{2} + \frac{\sin t}{2}\right)
\end{aligned}\right\} \text{2pts}$$

$= \dfrac{1}{2}(t\cos t + \sin t)$ 2pts

**Q:4** (12 points) Solve the initial value problem using Laplace transform $y'' - y' - 2y = \delta(t-\pi)$ with $y(0) = 1$, $y'(0) = 1$.

Let $F(s) = \mathcal{L}\{y\}$.

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{\delta(t-\pi)\}$$

$$s^2F(s) - sy(0) - y'(0) - [sF(s) - y(0)] - 2F(s) = e^{-\pi s} \quad \text{3pts}$$

$$(s^2 - s - 2)F(s) - s - 1 + 1 = e^{-\pi s}$$

$$F(s) = \frac{s}{s^2-s-2} + \frac{e^{-\pi s}}{s^2-s-2}. \quad \text{1 pt}$$

But, $$\frac{s}{s^2-s-2} = \frac{s}{(s+1)(s-2)} = \frac{\frac{1}{3}}{s+1} + \frac{\frac{2}{3}}{s-2} \quad \text{2pts}$$

$$\frac{1}{s^2-s-2} = \frac{1}{(s+1)(s-2)} = \frac{-\frac{1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} \quad \text{2pts}$$

Thus, $$F(s) = \frac{1}{3}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s-2} + \left[-\frac{1}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}\right]e^{-\pi s}$$

$$y(t) = \underbrace{\frac{1}{3}e^{-t} + \frac{2}{3}e^{2t}}_{\text{2pts}} + \underbrace{\left[-\frac{1}{3}e^{-(t-\pi)} + \frac{1}{3}e^{2(t-\pi)}\right]\mathcal{U}(t-\pi)}_{\text{2pts}}$$

**Q:5** (14 points) Show that the set of functions $\left\{\frac{\sqrt{2}}{2}, \cos\frac{n\pi}{2}x\right\}$, $n = 1, 2, 3, \cdots$ is orthogonal on $[0, 2]$. Also find norm of each function.

Let $w_0(x) = \frac{\sqrt{2}}{2}$, $w_n(x) = \cos\frac{n\pi}{2}x$

$$(w_0, w_n) = \frac{\sqrt{2}}{2}\int_0^2 \cos\frac{n\pi}{2}x\,dx = \frac{\sqrt{2}}{2}\left[\frac{\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right]_0^2 = \frac{\sqrt{2}}{n\pi}\sin n\pi = 0$$ 1 pt 2pts

$$(w_n, w_m) = \int_0^2 \cos\frac{n\pi}{2}x\cos\frac{m\pi}{2}x\,dx, \quad n \neq m$$ 1 pt

$$= \frac{1}{2}\int_0^2\left[\cos\frac{(n+m)\pi x}{2} + \cos\frac{(n-m)\pi x}{2}\right]dx$$ 1 pt

$$= \frac{1}{2}\left[\frac{\sin\frac{(n+m)\pi x}{2}}{\frac{(n+m)\pi}{2}} + \frac{\sin\frac{(n-m)\pi x}{2}}{\frac{(n-m)\pi}{2}}\right]_0^2$$

$$= \left[\frac{\sin(n+m)\pi}{n+m} + \frac{\sin(n-m)\pi}{(n-m)\pi}\right] = 0$$ 2pts

$$\|w_0\|^2 = \int_0^2\left(\frac{\sqrt{2}}{2}\right)^2 dx = \int_0^2 \frac{1}{2}dx = 1 \Rightarrow \|w_0\| = 1$$ 2pts 1pt

$$\|w_n\|^2 = \int_0^2 \cos^2\frac{n\pi}{2}x\,dx = \frac{1}{2}\int_0^2[1+\cos n\pi x]\,dx =$$ 1pt 1pt

$$= \frac{1}{2}\left[x + \frac{\sin n\pi x}{n\pi}\right]_0^2$$

$$= \frac{1}{2}\left[2 + \frac{\sin 2n\pi}{n\pi}\right] = 1$$ 1pt

$$\Rightarrow \|w_n\| = 1, \quad n = 1, 2, 3, \ldots$$ 1pt

**Q:6** (16 points) Find the Fourier series of the function $f(x) = \begin{cases} 1-x & -1 < x < 0 \\ 1 & 0 \le x < 1 \end{cases}$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x \quad , \text{ where} \quad (3\text{pts})$$

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1-x)\,dx + \int_{0}^{1} 1\,dx = \left[x - \frac{x^2}{2}\right]_{-1}^{0} + [x]_0^1 = \frac{5}{2} \quad (3\text{pts})$$

$$a_n = \int_{-1}^{1} f(x) \cos n\pi x\,dx = \int_{-1}^{0} (1-x)\cos n\pi x\,dx + \int_{0}^{1} \cos n\pi x\,dx \quad (2\text{pts})$$

$$\overset{(\text{Integ. by part})}{=} \left[\frac{(1-x)}{n\pi} \sin n\pi x\right]_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0} \sin n\pi x\,dx + \left[\frac{\sin n\pi x}{n\pi}\right]_0^1$$

$$= 0 + \frac{-1}{(n\pi)^2}\left[\cos n\pi x\right]_{-1}^{0} + 0$$

$$= -\frac{1}{(n\pi)^2}\left(1-(-1)^n\right) \quad (3\text{pts})$$

$$b_n = \int_{-1}^{1} f(x) \sin n\pi x\,dx = \int_{-1}^{0} (1-x)\sin n\pi x\,dx + \int_{0}^{1} \sin n\pi x\,dx \quad (2\text{pts})$$

$$\left(\text{by integration by part}\right) = \left[-\frac{(1-x)}{n\pi}\cos \pi n x\right]_{-1}^{0} - \frac{1}{n\pi}\int_{-1}^{0} \cos n\pi x\,dx - \left[\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= \frac{-1}{n\pi}\left(1 - 2\cos n\pi\right) - \frac{1}{(n\pi)^2}\left[\sin n\pi x\right]_{-1}^{0} - \frac{1}{n\pi}\left(\cos n\pi - 1\right)$$

$$= \frac{\cos n\pi}{n\pi} = \frac{(-1)^n}{n\pi} \quad (3\text{pts})$$

$$f(x) = \frac{5}{4} - \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}\left(1-(-1)^n\right)\cos \pi n x + \frac{(-1)^n}{n\pi} \sin n\pi x$$

**Q:7** (8+6 points) **(a)** Find the half-range Fourier cosine expansion of $f(x) = \sin 2x, \quad 0 \le x < \frac{\pi}{4}$.

**(b)** Show that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$.

a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 4nx$ , where (1pt)

$$a_0 = \frac{2}{\pi/4}\int_0^{\pi/4} f(x)\,dx = \frac{8}{\pi}\int_0^{\pi/4} \sin 2x\,dx = \frac{8}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/4} = \frac{4}{\pi}$$ (2pts)

$$a_n = \frac{2}{\pi/4}\int_0^{\pi/4} f(x)\cos 4nx\,dx = \frac{8}{\pi}\int_0^{\pi/4} \sin 2x \cos 4nx\,dx$$ (1pt)

$$= \frac{4}{\pi}\int_0^{\pi/4}\left[\sin(2+4n)x + \sin(2-4n)x\right]dx$$ (1pt)

$$= \frac{4}{\pi}\left[-\frac{\cos(2+4n)x}{2+4n} - \frac{\cos(2-4n)x}{2-4n}\right]_0^{\pi/4}$$

$$= \frac{4}{\pi}\left[\frac{-1}{2+4n}\left(\cos\left(\frac{\pi}{2}+n\pi\right)-1\right) - \frac{1}{2-4n}\left(\cos\left(\frac{\pi}{2}-n\pi\right)-1\right)\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{2+4n} + \frac{1}{2-4n}\right]$$ (3pts)

$$= \frac{4}{\pi}\,\frac{1}{1-4n^2}$$ (3pts)

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)}\cos 4nx$$

b) We set $x=0$,

$$f(0) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)}$$ (2 pts)

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$ (1 pt)

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}.$$ (3pts)