King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301   Major Exam 1
The First Semester of 2011-2012 (121)

Time Allowed: 120 Minutes

Name: ______________________ ID#: ______________

Instructor: ____________________ Sec #: ________ Serial #: ________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		14
2		12
3		14
4		16
5		12
6		16
7		16
Total		100

Q:1 (7+7 points) Given the position vector $\vec{r}(t) = 2\sqrt{2}\, t\, \hat{\imath} + e^{2t}\hat{\jmath} + e^{-2t}\hat{k}$ of a curve C:

(a) Find a unit tangent vector to the curve at $t = 0$.

(b) Find the length of the curve for $0 \le t \le 1$.

a) $r'(t)$ is a tangent vector, so that, $\vec{u} = \frac{r'(0)}{\|r'(0)\|}$ (2pts)

is a unit tangent vector to the curve at $t=0$.

$$r'(t) = 2\sqrt{2}\,\vec{i} + 2e^{2t}\vec{j} - 2e^{-2t}\vec{k} \longrightarrow \text{(1pt)}$$

$$r'(0) = 2\sqrt{2}\,\vec{i} + 2\vec{j} - 2\vec{k} \longrightarrow \text{(1pt)}$$

$$\|r'(0)\| = 2\sqrt{(\sqrt{2})^2 + 1 + 1} = 4 \longrightarrow \text{(1pt)}$$

$$\boxed{\vec{u} = \frac{\sqrt{2}}{2}\vec{i} + \frac{1}{2}\vec{j} - \frac{1}{2}\vec{k}} \longrightarrow \text{(2pts)}$$

b)

$$s = \int_0^1 \|r'(t)\|\,dt \longrightarrow \text{(2pts)}$$

$$= 2\int_0^1 \sqrt{2 + e^{4t} + e^{-4t}}\,dt \longrightarrow \text{(1pt)}$$

$$= 2\int_0^1 \sqrt{(e^{2t} + e^{-2t})^2}\,dt \longrightarrow \text{(1pt)}$$

$$\left.\begin{aligned} &= 2\int_0^1 (e^{2t} + e^{-2t})\,dt \\ &= \left[e^{2t} - e^{-2t}\right]_0^1 \\ &\boxed{s = e^2 - e^{-2}} \end{aligned}\right\} \text{(3pts)}$$

Q:2 (6+6 points) Let $f(x,y,z) = xy^2 + 2x^2y + z^2$.

(a) Find the directional derivative of $f(x,y,z)$ at $(-1,1,2)$ in the direction of $\hat{\imath} - \hat{\jmath} + 3\hat{k}$.

(b) Find the direction and value of maximum directional derivative of $f(x,y,z)$ at $(1,0,1)$.

a) $\nabla f(x,y,z) = (y^2+4xy)\vec{i} + (2xy+2x^2)\vec{j} + 2z\,\vec{k}$ 3pts

$\nabla f(-1,1,2) = -3\,\vec{i} + 4\,\vec{k}$ $\longrightarrow$ 1pt

The unit vector in the direction of $\vec{i}-\vec{j}+3\vec{k}$

is $\vec{u} = \frac{1}{\sqrt{1+1+9}}(\vec{i}-\vec{j}+3\vec{k})$

$= \frac{1}{\sqrt{11}}\vec{i} - \frac{1}{\sqrt{11}}\vec{j} + \frac{3}{\sqrt{11}}\vec{k}$ $\longrightarrow$ 1pt

The directional derivative of $f(x,y,z)$ at $(-1,1,2)$ in the direction of $\vec{i}-\vec{j}+3\vec{k}$ is

$D_{\vec{u}} f(-1,1,2) = \nabla f(-1,1,2)\cdot\vec{u}$

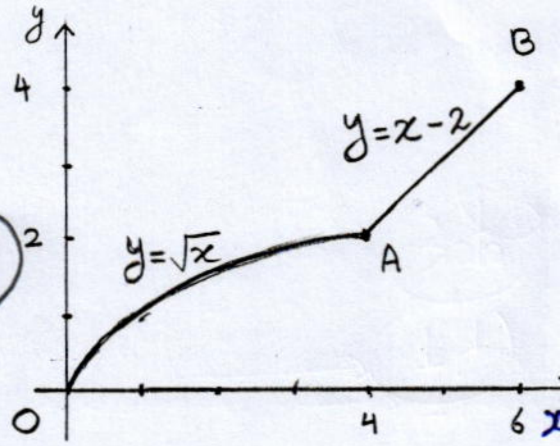
$= \frac{-3}{\sqrt{11}} + \frac{12}{\sqrt{11}}$

$\boxed{D_{\vec{u}} f(-1,1,2) = \frac{9}{11}\sqrt{11}}$ 1pt

b.) The direction of maximum directional derivative is $\nabla f(1,0,1) = 2\vec{j} + 2\vec{k}$. 3pts

The maximum directional derivative at $(1,0,1)$ is $\|\nabla f(1,0,1)\| = 2\sqrt{2}$. 3pts

Q:3 (a) (8 points) Evaluate the line integral $\int_C (x-3y)dx + ydy$ along the path from O to B as shown in the figure:



a) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where

$\mathcal{C}_1$: $0 \le x \le 4$, $y = \sqrt{x}$

$\mathcal{C}_2$: $4 \le x \le 6$, $y = x-2$

1pt

$$\int_{\mathcal{C}_1} (x-3y)\,dx + y\,dy = \int_0^4 (x-3\sqrt{x})\,dx + \sqrt{x}\,\frac{dx}{2\sqrt{x}}$$

3pts

$$= \int_0^4 \left(x - 3\sqrt{x} + \tfrac{1}{2}\right) dx = \left[\frac{x^2}{2} - 2x^{3/2} + \frac{x}{2}\right]_0^4 = -6$$

$$\int_{\mathcal{C}_2} (x-3y)\,dx + y\,dy = \int_4^6 [x-3(x-2)]\,dx + (x-2)\,dx$$

3pts

$$= \int_4^6 (-x+4)\,dx = \left[-\frac{x^2}{2} + 4x\right]_4^6 = -2$$

$$\Rightarrow \boxed{\int_{\mathcal{C}} (x-3y)\,dx + y\,dy = -8}$$

1pt

(b) (6 points) Evaluate the line integral $\int_C x^2\,dx + (x+z)\,dy + y^2\,dz$, where the path C is given by $x=t,\ y=t^2,\ z=-t,\ 0 \le t \le 1$.

$$\int_{\mathcal{C}} x^2 dx + (x+z)\,dy + y^2 dz = \int_0^1 t^2 dt + (t-t)\,2dt + t^4(-dt)$$

3pts

$$= \int_0^1 (t^2 - t^4)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{5}$$

$$= \frac{2}{15}$$

3pts

Q:4 (16 points) Determine if the vector field

$$\vec{F}(x,y) = (e^x \sin y - y\sin(xy))\,\hat{i} + (e^x \cos y - x\sin(xy))\,\hat{j}$$

is a conservative field. If so, find a potential function $\phi(x,y)$ for $\vec{F}$ and evaluate the integral $\int_C \vec{F}\cdot d\vec{r}$, where C is any path from $(1,\frac{\pi}{2})$ to $(2,\pi)$.

Let $P(x,y) = e^x \sin y - y\sin(xy)$

$Q(x,y) = e^x \cos y - x\sin(xy)$.

$$\left.\begin{cases} \dfrac{\partial Q}{\partial x} = e^x\cos y - \sin(xy) - xy\cos(xy) \\ \dfrac{\partial Q}{\partial y} = e^x\cos y - \sin(xy) - xy\cos(xy) \end{cases}\right\}$$ 2 pts

$Q_x = P_y \Rightarrow F(x,y)$ is conservative. 1 pt

there exists a potential $\phi(x,y)$ such that

$F = \nabla\phi$. 1 pt

$$\left.\begin{cases} \dfrac{\partial\phi}{\partial x} = e^x\sin y - y\sin(xy) & (1) \\ \dfrac{\partial\phi}{\partial y} = e^x\cos y - x\sin(xy) & (2) \end{cases}\right\}$$ 2 pts

Integrating (1), we find $\phi(x,y) = e^x\sin y + \cos(xy) + f(y)$ 2 pts

Substituting ϕ in (2), we find

$e^x\cos y - x\sin(xy) + f'(y) = e^x\cos y - x\sin xy$

$\Rightarrow f'(y) = 0 \Rightarrow f(y) = C$

$\Rightarrow \boxed{\phi(x,y) = e^x\sin y + \cos(xy) + C}$

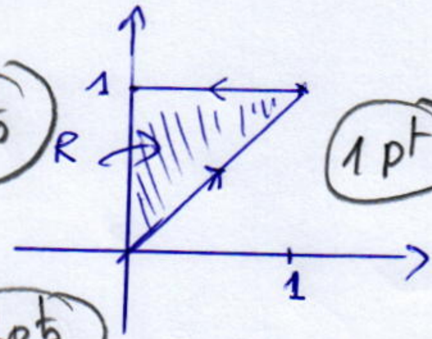
4 pts

$\int_{C} \vec{F}\cdot d\vec{r} = \phi(2,\pi) - \phi(1,\pi/2)$ 2 pts

$= 1 - e$ 2 pts

Q:5 (12 points) Use Green's theorem to evaluate the line integral $\oint_C 3e^{-x^2}dx + 2\ln(1+x^2)dy$, where C is the positively oriented triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$.

$$\int_C 3e^{-x^2}dx + 2\ln(1+x^2)dy = \iint_R \frac{4x}{1+x^2}\,dA$$ (4 pts)



(1 pt)

$$= \int_0^1\int_x^1 \frac{4x}{1+x^2}\,dy\,dx$$ (2 pts)

$$= \int_0^1 \frac{4x(1-x)}{1+x^2}\,dx$$

$$= \int_0^1 \left(\frac{4x}{1+x^2} - 4 + \frac{4}{1+x^2}\right) dx$$

$$= \left[2\ln(1+x^2) - 4x + 4\tan^{-1}x\right]_0^1$$

$$= 2\ln 2 - 4 + \pi$$ (5 pts)

or

$$\iint_R \frac{4x}{1+x^2}\,dA = \int_0^1\int_0^y \frac{4x}{1+x^2}\,dx\,dy$$ (2 pts)

$$= \int_0^1 2\ln(1+y^2)\,dy$$

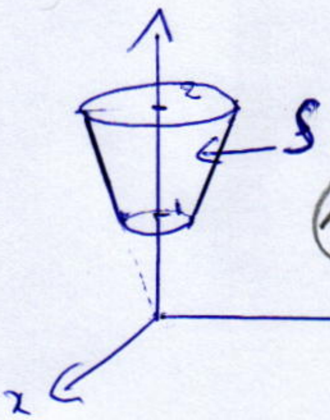
(Integration by part) $= \left[2y\ln(1+y^2)\right]_0^1 - 4\int_0^1 \frac{y^2}{1+y^2}\,dy$

$$= 2\ln 2 - 4\int_0^1 \left(1 - \frac{1}{1+y^2}\right) dy$$

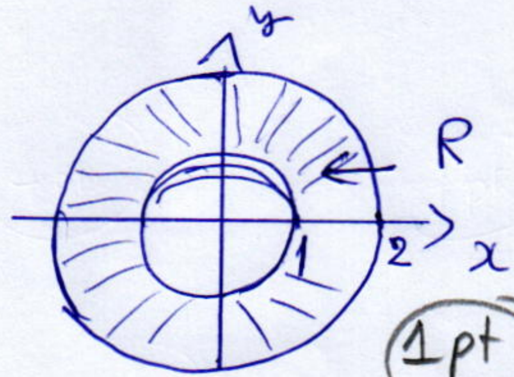
$$= 2\ln 2 - 4\left[y - \tan^{-1}y\right]_0^1$$

$$= 2\ln 2 - 4 + \pi.$$ (5 pts)

Q:6 (16 points) Evaluate the surface integral $\int\int_S y^2 z dS$, where S that portion of the cone $z = \sqrt{x^2+y^2}$ that lies between the planes $z = 1$ and $z = 2$.



1pt



1pt

Let $z = f(x,y) = \sqrt{x^2+y^2}$.

We have $f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$

$$\iint_S y^2 z\, dS = \iint_R y^2\sqrt{x^2+y^2}\sqrt{1+f_x^2+f_y^2}\, dA \quad \text{4pts}$$

$$1+f_x^2+f_y^2 = 1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2} = 2 \quad \text{2pts}$$

Thus, $\iint_S y^2 z\, dS = \sqrt{2}\iint_R y^2\sqrt{x^2+y^2}\, dA$ 1pt

Using polar coordinates,
$R = \{(r,\theta) / \ 0 \le \theta \le 2\pi, \quad 1 \le r \le 2\}$

So that, $\iint_S y^2 z\, dS = \sqrt{2}\int_0^{2\pi}\int_1^2 r^2\sin^2\theta \; r \; r\, dr\, d\theta$

$= \sqrt{2}\int_0^{2\pi} \sin^2\theta \left[\frac{r^5}{5}\right]_1^2 d\theta$

$= \frac{31}{5}\sqrt{2}\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta$

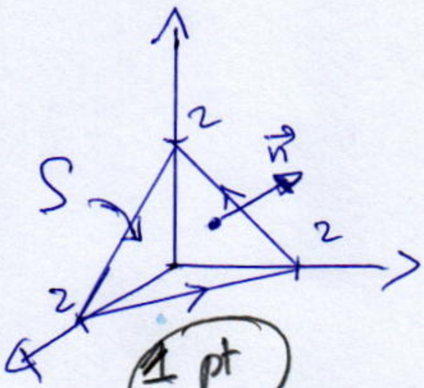
$= \frac{31}{5}\sqrt{2}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$

$$\boxed{\iint_S y^2 z\, dS = \frac{31\sqrt{2}}{5}\pi}$$

7pts

Q:7 (16 points) Use Stokes' theorem to evaluate the integral $\oint_C \vec{F} \cdot d\vec{r}$, where

$\vec{F}(x,y,z) = 2yz\ \hat{i} + xy\ \hat{j} + xy\ \hat{k}$ and C is the intersection of the plane $x+y+z=2$ with the coordinate planes in the first octant oriented positively.



1 pt

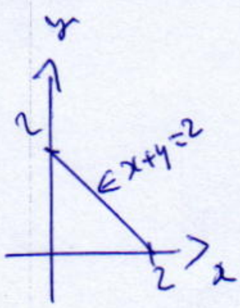
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, dS$$ 2pts

$S:\ z = f(x,y) = 2 - x - y$

$$\vec{n} = \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+f_x^2+f_y^2}} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$$ 2pts

$$\operatorname{curl} F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ 2yz & xy & xy \end{vmatrix} = x\vec{i} + y\vec{j} + (y-2z)\vec{k}$$ 3pts

$$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, dS = \frac{1}{\sqrt{3}}\iint_S (x+y+y-2z)\, dS$$



$$= \frac{1}{\sqrt{3}}\iint_R [x+2y-2(2-x-y)]\sqrt{1+f_x^2+f_y^2}\, dA$$

$$= \frac{1}{\sqrt{3}}\iint_R (3x+4y-4)\sqrt{3}\, dA$$

$$= \int_0^2\int_0^{2-x} (3x+4y-4)\, dy\, dx$$

$$= \int_0^2 \left[(3x-4)(2-x) + 2(2-x)^2\right] dx$$

$$= \int_0^2 x(2-x)\, dx = \left[x^2 - \frac{x^3}{3}\right]_0^2$$

8pts

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \frac{4}{3}.}$$